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Mathematical Reviews

Vol. 3, No. 1 JANUARY, 1942 Pages 1-32

ANALYSIS

Theory of Probability

von Mises, R. On the foundations of probability and statistics. Ann. Math. Statistics 12, 191–205 (1941). [MF 4710]

Doob, J. L. Probability as measure. Ann. Math. Statistics 12, 206–214 (1941). [MF 4711]

von Mises, R. and Doob, J. L. Discussion of papers on probability theory. Ann. Math. Statistics 12, 215–217 (1941). [MF 4712]

Kemble, Edwin C. The probability concept. Philos. Sci. 8, 204–232 (1941). [MF 4762]

van Dantzig, D. Mathematical and empirical foundations of the calculus of probability. Nederl. Tijdschr. Natuurkunde 8, 70–91, discussion, 91–93 (1941). (Dutch. English summary) [MF 4483]

Borel, Émile. Une objection à la définition empirique de la probabilité. C. R. Acad. Sci. Paris 211, 312–313 (1940). [MF 5351]

In cases such as the tossing of perfect coins, dice, etc., symmetry of figure allows one to assert exact equality of probabilities of the outcomes, whereas actual trials are excessively unlikely to give exactly the same number of heads as tails, etc. The author regards this as an argument against the empirical conception of probability. He evidently is willing in the definition by symmetry to ignore the lack of symmetry of the initial and attendant conditions of the toss. *B. O. Koopman* (New York, N. Y.).

Kendall, M. G. A theory of randomness. Biometrika 32, 1–15 (1941). [MF 3856]

In this paper the author presents some comments on a theory of randomness bearing some resemblance to von Mises' theory of Kollectivs. The principal departure from von Mises' theory lies in an attempt to supplant the idea of the irregular Kollectiv by considering infinite "proper suites" and aggregates of "selectors" to be applied to them. By a suite the author means an infinite sequence of "characteristics," each of which may be one of r symbols, say $A_1, A_2, \cdots, A_r$. By a proper suite K is meant one in which the proportion of A_i's in the first n elements of the suite tends to a limit as $n\to\infty$ for $i=1, 2, \cdots, r$. By a selector S is meant an infinite sequence of increasing positive integers. If those numbers in K are chosen whose ordinals are the numbers appearing in S, a derived suite SK is obtained. If the limiting frequency of A_i in SK is the same as for A_i in K, then K is said to be random for A_i with respect to S. Selector domains, statistical independence, convolution and local randomness are defined and briefly discussed. Several simple properties of derived suites are given, but the author does not go beyond his definitions to any significant extent. A considerable number of results, which are not referred to by the author, have been published by Copeland and Wald, dealing with the problem of the measure of selectors (represented, for example, as binary numbers) which preserve the limiting frequencies in the original suite, and related problems. Most of the paper is devoted to a general non-mathematical discussion of the notion of randomness in practical statistics from the point of view of the theory outlined at the beginning of the paper. *S. S. Wilks.*

Meier, J. Zur Theorie der unabhängigen Wahrscheinlichkeiten. Mitt. Verein. Schweiz. Versich.-Math. 39, 53–7? (1940). [MF 4421]

The theory of independent decrement probabilities, originated by Karup, is developed from a set of postulates. This set of postulates does not determine the probabilities uniquely, but numerical examples indicate that it restricts their values to very narrow intervals. *Z. W. Birnbaum.*

McMillan, Brockway. On two problems of sampling. Ann. of Math. (2) 42, 437–445 (1941). [MF 4293]

The author considers a large number of urns of various compositions of white and black balls. To each urn is assigned an a priori distribution giving the probability that $p<x$, where p is the probability of drawing a white ball. A sample is obtained by drawing one ball from each urn. On the basis of such a sample it is desired to estimate the a posteriori distribution for the average of the probabilities associated with the various urns, first on the assumption that the exact composition of the sample is known and second on the assumption that only the total number of white balls in the sample is known. The author shows that with suitable restrictions both of these distributions approach a definite asymptotic distribution as the number of urns is indefinitely increased. Bochner and von Mises considered the special case of this problem in which the a priori distributions were all the same. The author uses the more powerful method of Bochner which enables him to estimate the error incurred by using the asymptotic distribution for a finite sample. *A. H. Copeland* (Ann Arbor, Mich.).

Feldheim, E. Nouvelle démonstration et généralisation d'un théorème du calcul des probabilités dû à Simmons. J. Math. Pures Appl. (9) 20, 1–16 (1941). [MF 4636]

An Italian version of this paper was published in Giorn. Ist. Ital. Attuari 10, 229–243 (1939); cf. these Rev. 1, 246; a Hungarian version in Mat. Fiz. Lapok 45, 99–114 (1938).

Hadwiger, H. Bemerkung zum Problem des Ruins beim Spiele. Mitt. Verein. Schweiz. Versich.-Math. 40, 41–44 (1940). [MF 4390]

The author treats the classical problem of the ruin of a gambler with initial capital a, who has at each game the probability $\frac{1}{2}$ of winning α or of loosing β. The probability $p_n(a)$ that the gambler's capital will not fall below zero during n games satisfies a difference equation which can be approximated by a differential equation of the parabolic

type. Thus the author obtains approximate expressions for $p_n(a)$. This method is in line with the heuristic theory of Bachelier [in papers and books from 1900 onward; cf. his latest book, Lois des grands nombres, Paris, 1937]. [For results due to Petrowsky and Kolmogoroff, cf. Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Ergebnisse der Mathematik, vol. 2, no. 4, Berlin, 1933.]

W. Feller (Providence, R. I.).

Rosenblatt, Alfred. Sur le concept de contagion de M. G. Pólya dans le calcul del probabilités. Divers schèmes. Application à la peste bubonique au Pérou. Actas Acad. Ci. Lima **3**, 186–204 (1940). [MF 4360]

The author modifies the Pólya-Eggenberger scheme of urns [Z. Angew. Math. Mech. **3**, 279–289 (1923)] so that Δ red balls are added if a red ball was drawn, and Δ' black ones if a black ball was drawn. In the original scheme $\Delta=\Delta'$. The author carries through the necessary computations and finds the limiting distribution [cf. also a recent paper by Kitagawa, Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **1**, 167–194 (1941); these Rev. **2**, 230]. *W. Feller.*

Rosenblatt, Alfredo. On the law of large numbers in the theory of probability. Publ. Inst. Mat. Univ. Nac. Litoral **2**, 141–146 (1940). (Spanish) [MF 4234]

The author uses the inequality analogous to Tchebycheff's inequality, but involving moments of order k instead of order 2, and applies this inequality to obtain better limits for the N in the Bernoulli theorem that the usual average of n chance variables is within ϵ of its limit with probability not less than η if $n\geqq N$. *J. L. Doob* (Princeton, N. J.).

Baticle, Edgar. Sur la composition des probabilités de densités constantes. C. R. Acad. Sci. Paris **211**, 420–422 (1940). [MF 5356]

The author obtains a formula for the distribution of the sum of n independent random variables where each variable possesses a positive constant probability density throughout the interior of a given interval and zero density outside the interval. Thus the probability is 1 that the values taken on by the variables will be the coordinates of a point in an appropriate n-dimensional parallelepiped. If the sum of these coordinates is less than s, the point must lie in a portion which is cut off from this parallelepiped by a hyperplane. The ratio of this fractional volume to the total volume is the probability that the sum is less than s. The author gives a simple formula for this ratio.

A. H. Copeland (Ann Arbor, Mich.).

Dugué, Daniel. Sur quelques exemples de factorisation de variables aléatoires. C. R. Acad. Sci. Paris **212**, 838–840 (1941). [MF 5032]

This is a contribution to the so-called arithmetic of distribution functions. By an example the author shows that the formal product (convolution) of two distribution functions may be divisible by a Poisson distribution even if none of the factors contains a Poisson distribution. Furthermore the author points out the different possibilities of factoring the distribution function which is linear in $-\frac{1}{2}<x<\frac{1}{2}$ and constant elsewhere. *W. Feller* (Providence, R. I.).

Orts, J. M.[a] **The stability of the normal probability distribution for two random variables.** Revista Mat. Hisp.-Amer. (4) **1**, 34–36 (1941). (Spanish) [MF 4724]

Without proof the author remarks that Pólya's theorem [Math. Z. **18**, 96–108 (1923)] by which the normal distribution is the only stable distribution with finite second moment holds equally in two dimensions [for a proof cf., for example, Cramér, Random Variables, Cambridge Tracts no. 36, 1937, p. 111]. *W. Feller* (Providence, R. I.).

Steinhaus, H. Sur les fonctions indépendantes. VI. Studia Math. **9**, 121–132 (1940). (French. Ukrainian summary) [MF 5261]

Let $f(t)$, defined for $0\leqq t<\infty$, be measurable with respect to relative measure [Kac and Steinhaus, Studia Math. **7**, 1–15 (1938)], and let $\{f(t)\}=f(t)-[f(t)]$, where $[x]$ denotes the greatest integer in x. One says that $f(t)$ is uniformly distributed (mod 1) if for every $\lambda(0\leqq\lambda\leqq 1)$ the relative measure of the set of t's for which $\{f(t)\}<\lambda$ is equal to λ. The author proves several theorems showing the connection between statistical independence and uniform distribution (mod 1). For instance, if for every pair h, k of integers $(h^2+k^2>0)$ the function $hf(t)+kg(t)$ is uniformly distributed (mod 1), then $\{f(t)\}$ and $\{g(t)\}$ are statistically independent. As an application it is shown that for every pair a_1, a_2 of different real numbers the functions $\sin 2\pi(t+a_1)^2$ and $\sin 2\pi(t+a_2)^2$ are statistically independent. This last result (which gives an answer to a question of Kampé de Fériet) is included as a particular case in a more general investigation by Agnew and the reviewer [Bull. Amer. Math. Soc. **47**, 148–154 (1941); cf. these Rev. **2**, 229]. *M. Kac* (Ithaca, N. Y.).

Wintner, Aurel. On the iteration of distribution functions in the calculus of probability. Union Mat. Argentina, Publ. no. 18, 12 pp. (1941). (English. Spanish translation) [MF 4719]

Let $\Phi(x)$ be a distribution function whose second moment is infinite, and let $\Phi_n(x)$ be the nth iterated convolution of Φ with itself. The author proves that $\lim_{n\to\infty}[\Phi_n(x\sqrt{n})-\Phi_n(-x\sqrt{n})]=0$ for every x, using characteristic functions. Kac and Khintchine have also proved this result [C. R. Acad. Sci. Paris **202**, 1963–1965 (1936)], assuming in addition, however, the finiteness of the first moment of Φ.

J. L. Doob (Princeton, N. J.).

Groshev, A. Sur le domaine d'attraction de la loi de Poisson. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 165–172 (1941). (Russian. French summary) [MF 4513]

Theorem I gives a necessary and sufficient condition that an arbitrary distribution function, and Theorem II that an infinitely divisible one, belongs to the domain of partial attraction of Poisson's law. The results and methods of proof are related to the ones of Gnedenko reviewed previously [these Rev. **1**, 341; **2**, 228]. *M. Kac.*

Hartman, Philip. Normal distributions and the law of the iterated logarithm. Amer. J. Math. **63**, 584–588 (1941). [MF 4682]

Let the x_n be mutually independent normally distributed stochastic variables with mean 0 and variances b_n. Let $s_n=x_1+\cdots+x_n$ and $B_n=b_1+\cdots+b_n$ and suppose that $B_n\to\infty$. The author proves the following theorem which is, for this particular class of sequences, a generalization of the law of the iterated logarithm. Let α be the number satisfying the requirements: (i) if $r>\alpha$ and if $\{n_k\}$ is any sequence of integers for which there exists a $c>1$ such that $B_{n_k}>cB_{n_{k-1}}$, then $\sum|\log B_{n_k}|^{-r}$ converges; (ii) if $r<\alpha$ then there exists a sequence $\{n_k\}$ and a $c>1$ such that

$B_{n_k} > cB_{n_{k-1}}$ and for which $\sum|\log B_{n_k}|^{-r}$ diverges. Then $\overline{\lim}\, s_n/(2B_n \log\log B_n)^{\frac{1}{2}} = \alpha^{\frac{1}{2}}$ with probability 1. It is shown that actually it suffices to consider a single sequence $\{n_k\}$ defined by induction so that $B_{n_k-1} \leqq CB_{n_{k-1}}$ and $B_{n_k} > CB_{n_{k-1}}$, where $C>1$ is arbitrarily fixed. *W. Feller.*

Loève, Michel. Sur les systèmes d'événements; application à deux théorèmes classiques. C. R. Acad. Sci. Paris **212**, 261–263 (1941). [MF 4898]

If event B_i implies event A_i, then

$$\begin{aligned}\Pr[(A_1-B_1)(A_2-B_2)\cdots(A_m-B_m)] &= \Pr[A_1A_2\cdots A_m]+\cdots \\ &+(-1)^k\sum\Pr(B_{i_1}\cdots B_{i_k}A_{i_{k+1}}\cdots A_{i_m})+\cdots \\ &+(-1)^m\Pr(B_{i_1}B_{i_2}\cdots B_{i_m}).\end{aligned}$$

In analogy with the method of King, Loève expresses the probability that exactly r of m events will occur in the symbolic form $M^r e^{-M}/r!$, where M^r has been substituted for the rth factorial moment $M_{(r)}$ and e^{-M} is to be given its usual Maclaurin expansion. In case the distribution is that of Poisson, the expression interpreted algebraically, instead of symbolically, gives the required probability. A generating function $G(u)=e^{M(u-1)}$ is found. For two sets of events the corresponding symbolic expression is $M^rM'^se^{-(M+M')}/r!s!$, resembling the Poisson law in two dimensions. Application is made to the game of rencontre. Inequalities are obtained which are generalizations of certain inequalities of Fréchet and Gumbel. For an enumerable set of events, the author considers the probability that an infinity of events will occur, and generalizes theorems of Borel and Borel-Cantelli. *E. L. Dodd* (Austin, Tex.).

Loève, Michel. La loi des grands nombres pour des variables aléatoires liées et des événements liés. C. R. Acad. Sci. Paris **212**, 840–843 (1941). [MF 5033]

Theorems of the following type are stated without proof. Let $X_1, X_2, \cdots$ be a sequence of chance variables, and let $\{a_n\}$ be a sequence of constants. Denote by $\mathfrak{M}(X)$ the expectation of the chance variable X, and by $\mathfrak{M}'(X_i)$ the conditional expectation of X_i for given values of $X_1, \cdots, X_{i-1}$, a function of the latter variables. Then if

$$\lim_{n\to\infty}(1/a_n)\sum_1^n \sup|\mathfrak{M}'(X_i)|=0$$

and if

$$\lim_{n\to\infty}(1/a_n^2)\sum_1^n \mathfrak{M}(X_i^2)=0,$$

it follows that

$$\lim_{n\to\infty}\{(1/a_n)\sum_1^n X_j-(1/a_n)\sum_1^n\mathfrak{M}(X_j)\}=0.$$

This and other results stated generalize well-known results for mutually independent chance variables. *J. L. Doob.*

Beboutoff, M. Markoff chains with a compact state space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 482–483 (1941). [MF 4463]

The author investigates the analogues for Markoff processes of Kryloff-Bogoliouboff theorems on flows [Ann. of Math. (2) **38**, 65–113 (1937)]. [Proofs are omitted.] Let $P(E|x)$ be the probability of going from x (in a compact metric space S) to the set E. Besides the usual hypotheses on $P(E|x)$, it is also supposed that, for any continuous function $\varphi(x)$, $\int_S\varphi(y)P(dy|x)$ is continuous in x. He finds that there is an absolute probability function, and that there is the usual decomposition of the state space S into ergodic parts, etc., obtaining, however, a slightly more precise description than is usual of the non-dissipative part of S. *J. L. Doob* (Princeton, N. J.).

Doblin, W. Éléments d'une théorie générale des chaînes simples constantes de Markoff. Ann. École Norm. (3) **57**, 61–111 (1940). [MF 4633]

Let $P^{(1)}(x, E)$ be the conditional probability function of a Markoff chain: the probability of going from the point x into the set E in one transition. The set E varies over a Borel field of sets in an abstract space. Then $P^{(n)}(x, E)$, the probability of going from x into E in n transitions, is determined. The author studies the actual motion of the system applying the results to the study of the asymptotic properties of $P^{(n)}(x, E)$, $n\to\infty$. An analysis of the properties of sets E in their relation to the movement (invariance, etc.) is made, too detailed even to be summarized here, and general theorems are obtained which reduce to well-known results on Markoff chains when further hypotheses are made on $P^{(1)}(x, E)$. *J. L. Doob* (Princeton, N. J.).

Doubrowsky, V. Sur un problème limite de la théorie des probabilités. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 411–416 (1940). (Russian. French summary) [MF 4082]

Let a stochastic process have the property that every finite time interval (t, τ) can be subdivided into a finite number of time intervals in which the state of the considered system remains unchanged. Let x denote a particular state of the system and A the set of all possible states. It is assumed that a Borel field of sets is defined in A. Let $p(t, x, \tau)$ $(\tau>t)$ denote the probability that the system, which at the time t was in state x, will undergo a change before the time τ, and $P(t, x, E)$ the conditional probability that, if the system is at time t in the state x and undergoes at this time a change, it will assume a new state belonging to the subset E of A. Furthermore, let $p(t, x, \tau)=p(t, x)(\tau-t)+o(\tau-t)$, where the o symbol holds uniformly in t for $|t|<T$, and $p(t, x)$ is a bounded function, Riemann integrable with respect to t and measurable with respect to x. The author solves the problem of determining the probability $\omega(t, x, \tau)$ that the system which at time t was in state x will assume for some $s(t<s<\tau)$ a state belonging to a subset $e(s)$ of A. It is understood that $p(t, x)$ and $P(t, x, E)$ are known and that $e(s)$ is a given time-depending family of subsets of A. *M. Kac* (Ithaca, N. Y.).

Bachelier, Louis. Probabilités des oscillations maxima. C. R. Acad. Sci. Paris **212**, 836–838 (1941). [MF 5031]

The author continues his researches on differential stochastic processes in which the chance variable x_t-x_0 has a Gaussian distribution with mean 0 and variance depending on t, h. His previous results are summarized in his Nouvelles Méthodes des Calcul des Probabilités, Paris, 1939. Proofs are omitted. He gives, for example, the probability (density) that $\text{L.U.B.}_{0\le t\le T}(x_t-x_0)=x$, that the L.U.B. be attained at s and that $x(T)=z$, as a function of x, z, s. The advance over previous results lies in the specification of the point s where the L.U.B. is attained. *J. L. Doob.*

Ginsbourg, G. Sur les conditions suffisantes pour l'unicité des distributions limites. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 295–297 (1941). [MF 4383]

Ginsbourg, G. Sur les lois limites des distributions dans les procédés stochastiques. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 65–73 (1940). (Russian. French summary) [MF 4749]

The author studies distribution functions defined by the stochastic equations

$$\Delta y = A(y, t, (\Delta t)^{\frac{1}{2}})\Delta t + f(\alpha, y, t, (\Delta t)^{\frac{1}{2}})(\Delta t)^{\frac{1}{2}},$$

and the limits of these functions as $t\rightarrow\infty$. The methods and results depend on the general theory of S. Bernstein [Trav. Inst. Phys.-Math. Stekloff **5**, 95–123 (1934)].

M. Kac (Ithaca, N. Y.).

Fortet, Robert. Sur des fonctions aléatoires définies par leurs équations aux dérivées partielles. C. R. Acad. Sci. Paris **212**, 325–326 (1941). [MF 4904]

It is known that the stochastic variable $X(t)$ of a Markoff process is determined by its transition probability $F(t, x; \tau, \xi)$ which is the conditional probability that $X(\tau)\leqq\xi$ when it is known that $X(t)=x(t<\tau)$. For a certain class of processes, $F(t, x; \tau, \xi)$ is, as a function of t, x, a solution of the parabolic equation (*) $u_t+au_{xx}+bu_x=0$ [Kolmogoroff, Math. Ann. **104**, 415–458 (1931)]. Conversely, any such equation in general determines uniquely a process $X(t)$ [Feller, Math. Ann. **113**, 113–160 (1936)]. The author completes these investigations by a study of properties of $X(t)$ itself. He states that, under some very general conditions on a and b, $X(t)$ is continuous with probability one, and for sufficiently small $|t'-t''|$ and any constant $c>\sqrt{2}$ one has with probability one

$$|X(t'')-X(t')|\leqq c|(t''-t')\log(t''-t')|^{\frac{1}{2}}.$$

Turning to the generalized problem of the ruin the author states that there exists a probability $\phi(t, x, \tau)$ that, if $X(t)=x$, one shall have $X(t')=\psi(t')$ for at least one t' with $t<t'<\tau$, where $\psi(t)$ is any given differentiable function and $\psi(t)>x$. As a function of t, x this $\phi(t, x, \tau)$ is again a solution of (*) and is uniquely determined by some boundary conditions. The note does not contain proofs. *W. Feller.*

Kolmogoroff, A. N. Über das logarithmisch normale Verteilungsgesetz der Dimensionen der Teilchen bei Zerstückelung. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 99–101 (1941). [MF 4692]

It is stated that observations show that the logarithms of the sizes of particles such as mineral grains are frequently normally distributed [Rasumovski, in the same C. R. **28**, 814–816 (1940)]. The purpose of the present paper is to explain this phenomenon by a plausible probabilistic scheme. Consider a random process in which the number of particles at time t $(=0, 1, 2, \cdots)$ is $N(t)$, and the number of particles of dimension not greater than r is $N(r, t)$ (it is irrelevant how this "dimension" is defined). It is supposed that the probability that a particle of size r splits, during $(t, t+1)$, into n particles of sizes $x_1r, x_2r, \cdots, x_nr$ is independent of t and r. Let then $Q(x)$ be the mean value of the number of particles of size not greater than xr originated during $(t, t+1)$ from a particle of size r. It is shown that under some slight additional assumptions $N(e^x, t)/N(t)$ tends to a Gaussian distribution with mean value $mt=t\int_0^1\log y\,dQ(y)/Q(1)$ and variance $t\int_0^1(\log y-m)^2dQ(y)/Q(1)$. *W. Feller.*

Kolmogoroff, A. Interpolation und Extrapolation von stationären zufälligen Folgen. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 3–14 (1941). (Russian. German summary) [MF 4267]

Let t assume integer values only $(-\infty<t<\infty)$ and let $x(t)$ be a sequence of stochastic variables such that $E[x(t)]=0$ and $B(k)=E[x(t+k)x(t)]$ exists and is independent of t (stationary process in the sense of Khintchine). Given $n>0$ and $m\geqq 0$, the problem of linear extrapolation consists in finding constants $a_1, \cdots, a_n$ such that $L_e=a_1x(t-1)+\cdots+a_nx(t-n)$ gives a best approximation to $x(t+m)$, that is, that $\sigma^2=E[\{x(t+m)-L_e\}^2]$ becomes minimum. This minimum will be denoted by $\sigma_e^2(n, m)$. Obviously $\sigma_e^2(n, m)\downarrow\sigma_e^2(m)$ as $n\rightarrow\infty$. Similarly the (symmetric) linear interpolation requires the determination of constants $b_{-n}, b_{-n+1}, \cdots, b_{-1}, b_1, \cdots, b_n$ such that

$$L_i=b_{-n}x(t-n)+\cdots+b_{-1}x(t-1)+b_1x(t+1)+\cdots+b_nx(t+n)$$

approximates $x(t)$, that is, that $E[\{x(t)-L_i\}^2]$ becomes minimum. Again, as $n\rightarrow\infty$ the minimum $\sigma_i^2(n)$ of this expression tends to a limit σ_i^2. The author proves the following three theorems: (i)

$$W(\lambda)=B(0)\lambda+2\sum_1^\infty(B(k)/k)\sin k\lambda$$

is non-decreasing and

$$B(k)=(1/\pi)\int_0^\pi\cos k\lambda\,dW(\lambda).$$

(ii) Put $$P=(1/\pi)\int_0^\pi\log W'(\lambda)d\lambda.$$

If $P=-\infty$, then $\sigma_e^2(m)=0$. If P is finite, then $\sigma_e^2(m)=e^P(1+r_1^2+\cdots+r_m^2)$, where the r_k are determined from

$$\log\sum_0^\infty r_k\zeta^k\equiv\sum_1^\infty c_k\zeta^k,\quad c_k=(1/\pi)\int_0^\pi\cos k\lambda\log W'(\lambda)d\lambda.$$

(iii) $$\sigma_i^{-2}=(1/\pi)\int_0^\pi d\lambda/W'(\lambda)$$

(in particular $\sigma_i^2=0$ if the integral diverges).

These theorems follow from general results on stationary sequences of elements of the complex Hilbert space [Bull. Math. Univ. Moscou **2**, no. 6 (1941)]. Theorem 1 is due to H. Wold [Analysis of stationary time series, Uppsala, 1938]; the other theorems throw new light on Wold's results. [Cf. also the paper reviewed below.] *W. Feller.*

Kosulajeff, P. A. Sur les problèmes d'interpolation et d'extrapolation des suites stationnaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 13–17 (1941). [MF 4259]

The author treats the problems stated in the preceding review in the special case where there exists an integer N such that $B(k)=(1-|k|)/N$ for $|k|<N$ and $B(k)=0$ otherwise. One is led to such sequences by considering the moving average

$$x(t)=\{y(t)+y(t-1)+\cdots+y(t-N-1)\}/\sqrt{N}$$

of any sequence of stochastic variables $y(t)$ with $E[y(t)]=0$, $E[y^2(t)]=1$ and $E[y(t)y(t+\tau)]=0$ for $\tau\neq 0$. The author solves the problem by elementary computations and finds the values of $\sigma_e^2(n, 0)$ and of $\sigma_i^2(n)$. *W. Feller.*

Borel, Émile. Applications du calcul des probabilités aux problèmes concernant les nombres premiers. Théorème de Goldbach. C. R. Acad. Sci. Paris **212**, 317–320 (1941). [MF 4901]

The author proves a result from which he deduces that

the probability that the statement "every even number $2a$ is the sum of two primes less than $50(\log a)^{2}$" is false is less than 10^{-60}. *P. Erdős* (Philadelphia, Pa.).

Ornstein, L. S. and Milatz, J. M. W. The analogy between the statistics of numbers and statistical mechanics. Nederl. Akad. Wetensch., Proc. **44**, 163–172 (1941). [MF 4392]

The authors attempt to treat the problem of the frequency of digits in binary and decimal expansions by methods similar to the ones used by Gibbs in statistical mechanics. The article has only an expository value since most of the results are well known. Also the authors seem to be under a misapprehension that they have proved in a simple way the classical theorem of Borel to the effect that the binary expansion of almost every number has asymptotically equally many 0's and 1's. The misunderstanding is due to the confusion of the weak with the strong law of large numbers. *M. Kac* (Ithaca, N. Y.).

Santaló, L. A. A system of mean values in the theory of geometric probabilities. Revista Ci., Lima **43**, 147–154 (1941). (Spanish) [MF 4845]

Let L be the perimeter, F the surface of a convex domain K. We suppose K of white color and cover it arbitrarily with a black stripe of breadth Δ bounded by two parallel straight lines, then arbitrarily with a white stripe of the same breadth Δ, then again with a black one, and so on. The author defines and investigates by means of integral-geometrical methods the mean value $\bar{f}_n$ of the surface of K that is black after the nth step. He finds the following formulas:

$$\bar{f}_{2n+1}=\frac{\pi\Delta F+L\bar{f}_{2n}}{L+\pi\Delta},\quad \bar{f}_{2n+2}=\frac{L\bar{f}_{2n+1}}{L+\pi\Delta},$$

which permit the computation of $\bar{f}_n$. *P. Scherk.*

Theoretical Statistics

***Buros, Oscar Krisen, ed. The Second Yearbook of Research and Statistical Methodology Books and Reviews.** Gryphon Press, Highland Park, N. J., 1941. xx+383 pp. $5.00.

This book contains 1652 excerpts of book reviews from 283 journals published in English. It covers the fields of statistics and related subjects such as mathematical economics, population studies and general histories of science. It contains reviews published since 1938 and is the continuation of "Research and Statistical Methodology Books and Reviews of 1933–1938," by the same author.

***Tables of Probability Functions. Vol. I.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 65-2-97-33; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1941. xxviii+302 pp. $2.00.

Let

$$H(x)=(2/\pi^{-\frac{1}{2}})\int_0^x e^{-\alpha^2}d\alpha.$$

The main table of the present volume gives $H'(x)$ and $H(x)$ to 15 decimal places for $0\leqq x\leqq 1$ in steps of .0001, and for larger x in steps of .001. A second table gives $H'(x)$ and $1-H(x)$ to eight significant figures for $4\leqq x\leqq 10$ in steps of .01. It is unnecessary to point out that, as far as range and subdivision are concerned, these tables by far outrange all existing tables. They are to be followed by similar tables for the normalized probability functions. The completed work and the recent tables published by the British Association for the Advancement of Science [Cambridge, 1939; these Rev. **2**, 108] will complement each other in the best way and probably fill all practical needs. *W. Feller.*

Curtiss, J. H. Generating functions in the theory of statistics. Amer. Math. Monthly **48**, 374–386 (1941). [MF 5087]

This paper consists of an elementary expository treatment of the theory and applications of moment and semi-invariant generating functions in theoretical statistics.

From the introduction.

Steffensen, J. F. On the ω test of dependence between statistical variables. Skand. Aktuarietidskr. **1941**, 13–33 (1941). [MF 5083]

In Biometrika **26**, 251–255 (1934), the author proposed ω as a measure of dependence between two discrete variables x and y, where

$$\omega=\frac{2\overline{\sum}\,(p_{ij}-\sum_j p_{ij}\sum_i p_{ij})}{\overline{\sum}\,(p_{ij}-\sum_j p_{ij}\sum_i p_{ij})+1-\overline{\sum}\,p_{ij}^2},$$

where p_{ij} is the probability that x assumes the value x_i and y the value y_j, and $\overline{\sum}$ denotes summation for all i and j for which $p_{ij}>\sum_j p_{ij}\sum_i p_{ij}$. It was at that time proved that: (a) ω is always comprised between 0 and 1, (b) ω vanishes if and only if the variables are completely independent, and (c) $\omega=1$ if and only if there is complete dependence. The present paper shows that these same three properties of ω apply, in all essentials, for continuous variables. A short table is given to indicate how the correlation coefficient ρ is related to ω in the special case of normal correlation. For most of the range in ρ between 0 and 1, it is shown that ω is somewhat less than $\frac{2}{3}$ of ρ. *W. A. Shewhart.*

Steffensen, J. F. On the coefficient of correlation for continuous distributions. Skand. Aktuarietidskr. **1941**, 1–12 (1941). [MF 5082]

It is first shown that the coefficient of correlation ρ for two normally distributed statistical variables x and y possesses the following characteristics: (a) $\rho=0$ is necessary and sufficient in order that the variables be mutually independent; (b) ρ is always comprised between the limits $+1$ and -1; (c) that ρ approach $+1$ or -1 is necessary and sufficient in order that the distribution shall tend to complete dependence between the variables. It is then shown that the coefficient of correlation ρ for two variables x and y with distribution function $f(x, y)=Kf_1(\xi)f_2(\eta)$ possesses these same three characteristics, where $\xi=a_1x+b_1y$ and $\eta=a_2x+b_2y$ and where $f_1(\xi)$ and $f_2(\eta)$ may be any two continuous distributions not necessarily normal, for instance, Pearson types. *W. A. Shewhart* (New York, N. Y.).

Höffding, Wassilij. Maszstabinvariante Korrelationstheorie. Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin **5**, 181–233 (1940). [MF 4505]

The present paper is an expository treatment of a problem in elementary mathematical statistics. Suppose $\omega(\xi, \eta)$ is a probability density function defined over the entire $\xi\eta$

plane. Let $u(\xi)$ and $v(\eta)$ be the probability density functions of ξ and η taken separately (marginal distributions). If the probability element $\omega(\xi, \eta)d\xi d\eta$ is subjected to the transformation

$$x+\tfrac{1}{2}=\int_{-\infty}^{\xi} u(\xi)d\xi, \quad y+\tfrac{1}{2}=\int_{-\infty}^{\eta} v(\eta)d\eta,$$

a probability element $s(x, y)dydx$ is obtained, where the probability density function is identically zero outside the square $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$, $-\frac{1}{2}\leqq y\leqq\frac{1}{2}$ and is such that the probability density functions of the marginal distributions of x and y are unity over the ranges $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$ and $-\frac{1}{2}\leqq y\leqq\frac{1}{2}$, respectively, and zero otherwise. In case ξ and η are independent in the probability sense, $s(x, y)\equiv 1$ over the square $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$, $-\frac{1}{2}\leqq y\leqq\frac{1}{2}$ and zero otherwise. The author discusses the joint moments of x and y for the distribution function $s(x, y)$ and also for the cumulative distribution function

$$S(x, y)=\int_{-\frac{1}{2}}^{y}\int_{-\frac{1}{2}}^{x} s(x, y)dydx,$$

giving formulas in the case of independence. In case of independence $S(x, y)\equiv(x+\frac{1}{2})(y+\frac{1}{2})$. The mean values

$$E[(s(x, y)-1)^2]=\phi^2, \quad E[(S(x, y)-(x+\tfrac{1}{2})(y+\tfrac{1}{2}))^2]=\Phi^2,$$

under the assumption of independence, are proposed as indices for measuring departure from independence. The author devotes a section to the problem of expanding $s(x, y)$ and $S(x, y)$ into series of the form $\sum_{i,j=0}^{\infty} a_{ij}Q_i(x)Q_j(x)$, where $Q(x)$ is a Legendre polynomial of degree i. Finally the theory is applied to the case in which $\omega(\zeta, \eta)$ is a bivariate normal distribution and expressions are found for ϕ^2, Φ^2 and for the correlation ρ between x and y. The expressions for ϕ^2 and ρ are the well-known Pearson expressions $r^2/(1-r^2)$ and $(6/\pi)\sin^{-1}(r/2)$, respectively, where r is the correlation coefficient between ζ and η. *S. S. Wilks.*

Obuchoff, A. M. Eine Korrelationstheorie der Vektoren. Uchenye Zapiski Moskov. Gos. Univ. Matematika **45**, 73–92 (1940). (Russian. German summary) [MF 3733]

A systematic presentation of the theory of correlation between random vectors by means of tensor calculus, as initiated by Hotelling [Biometrika **28**, 321 (1936)] and also by Obuchoff [Izvestia Akad. Nauk SSSR **3** (1938)]. New developments include concepts of tensors of regression and of relative variances. *J. Neyman* (Berkeley, Calif.).

Masuyama, Motosaburô. The totally orthonormalised vector set and the normal form of correlation tensor. Proc. Phys.-Math. Soc. Japan (3) **23**, 346–351 (1941). [MF 4769]

Given m linearly independent n-dimensional vector functions, the author first constructs from them a set of "totally" orthonormal vector functions, and then using these he determines the "regression tensor" coefficients for the regression of an arbitrary vector function on this set. For two vector functions, totally normalized, the "normal correlation tensor" is defined and briefly discussed. *C. C. Craig.*

Masuyama, Motosaburô. The mean angle between two vector sets. Proc. Phys.-Math. Soc. Japan (3) **23**, 351–355 (1941). [MF 4770]

The author discusses the formula of Dietzius [Meteorol. Z. **32**, 433 (1915)], pointing out instances in which it gives a mean angle between two vector sets which has no physical meaning. He points out the necessary and sufficient condition for the existence of a definite angle between two linearly related vector functions and suggests that the corresponding stochastic condition be applied to stochastically related vector functions. *C. C. Craig.*

Masuyama, Motosaburô. The standard error of the mean vector. Proc. Phys.-Math. Soc. Japan (3) **23**, 194–195 (1941). [MF 4520]

Using the ordinary results for the standard error of a mean, the standard error of a mean vector is set down together with its representation in tensor form. There is some discussion, particularly of a geometrical interpretation of the result. *C. C. Craig* (Ann Arbor, Mich.).

Masuyama, Motosaburô. The normal law of frequency for vector quantities. Proc. Phys.-Math. Soc. Japan (3) **23**, 196–199 (1941). [MF 4521]

It is pointed out that the n-dimensional normal frequency function is also the normal frequency function for an n-dimensional vector. Then the normal frequency function for two or more such vectors is written out and the meaning of independence of two or more vectors is discussed in this connection. *C. C. Craig* (Ann Arbor, Mich.).

Masuyama, Motosaburô. On the characteristic values of the correlation tensor and a new measure of correlation between vector quantities. Proc. Phys.-Math. Soc. Japan (3) **23**, 199–204 (1941). [MF 4522]

Eight conditions to be satisfied by a measure of the correlation between vector quantities are set down. Then, if $g(t)$ and $u(t)$ are n-dimensional vectors, the trace of $[gu]\cdot[uu]^{-1}\cdot[ug]\cdot[gg]^{-1}$ divided by n, designated by $\rho(g;u)$, is proposed as such a measure. The way in which the conditions are fulfilled is discussed. In particular, if $g(t)$ is a general function of m vectors $u_i(t)$, then $\sum_1^m \rho(g;u_i)=1$, and the name "additive correlation coefficient" is proposed. If $n=1$, $\rho(g;u)$ is the square of the ordinary Pearsonian coefficient. *C. C. Craig* (Ann Arbor, Mich.).

Kavanagh, Arthur J. Note on the adjustment of observations. Ann. Math. Statistics **12**, 111–114 (1941). [MF 4009]

The need is pointed out for a generalization of the method of least squares such that, instead of the sum of squares of the deviations, a quadratic form containing also cross-products is made minimum. A procedure for solving problems of this kind, also with side conditions, is outlined.

Z. W. Birnbaum (Seattle, Wash.).

Bachmann, W. K. L'ellipsoïde d'erreur. Schweiz. Z. Vermessgswes. **28**, 181–197, 201–208, 213–216 (1940). [MF 3714]

A systematic presentation of the theory of ellipses of error, including theorems on decomposition and addition of such ellipses, and arguments leading to graphical and numerical methods. The second paper contains a generalization of the theory to the n-dimensional case, and the third paper a geometrical interpretation and applications.

Z. W. Birnbaum (Seattle, Wash.).

Feldheim, E. Sul rapporto fra la media dei quadrati di più errori e il quadrato della media dei loro valori assoluti. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 296–305 (1940). [MF 4780]

Let $X_1, X_2, \cdots, X_n$ be n independent variates, each following the same normal law with mean equal to zero and a S.E.

equal to σ. Let further $U=(\sum_{i=1}^n X_i^2)/n$, $Y=(\sum_{i=1}^n |X_i|)/n$, and $Z=U/Y^2$. The purpose of the paper is to deduce the expectation of Z. The author notices that the random variables U and Z are mutually independent and, after some calculations, arrives at the obvious relation between the expectations of the three variables: $E(Y^{-2})=E(Z)E(U^{-1})$. If $n\geqq 3$, the familiar χ^2 distribution gives $E(U^{-1})=n/\sigma^2(n-2)$, and so $E(Z)=\sigma^2(n-2)E(Y^{-2})/n$. Further calculations of the author, aiming at numerical values of $E(Z)$, are based on approximate evaluation of $E(Y^{-2})$. This is obtained by using the known two first moments of Y to fit a Type III Pearson curve, which is then considered as an approximation to the actual distribution of Y. *J. Neyman.*

von Neumann, J., Kent, R. H., Bellinson, H. R. and Hart, B. I. The mean square successive difference. Ann. Math. Statistics 12, 153–162 (1941). [MF 4708]

Consider a sample of size n in which the observed values are $x_1, x_2, \cdots, x_n$, and let $(n-1)\delta^2=\sum_1^{n-1}(x_{i+1}-x_i)^2$. The authors have studied the sampling theory of δ^2 assuming the elements in the sample to have been independently drawn from a normal population with known variance σ^2. The first four moments of δ^2 are determined, and a Pearson Type VI curve is fitted to the distribution of δ^2/σ^2. The values of the constants for the fitted curves are given for $n=5, 7, 10, 15, 20, 25, 50$ and a comparison is made in each case between the true β_2 and the β_2 of the fitted curve.
S. S. Wilks (Princeton, N. J.).

Williams, J. D. Moments of the ratio of the mean square successive difference to the mean square difference in samples from a normal universe. Ann. Math. Statistics 12, 239–241 (1941). [MF 4718]

Given an ordered sample $x_1, x_2, \cdots, x_n$ from a normal population with zero mean, the author derives the moment generating function for the sample sum of squares and the quantity defined by $(n-1)\delta^2=\sum_{j=1}^{n-1}(k_j-k_{j+1})^2$. From this by a neat device he finds the moment generating function for the ratio of these two quantities, from which it appears that the moments of this ratio is the ratio of the corresponding moments of the numerator and the denominator. The first four moments of this ratio are written out, from which it is easy to get the same moments of δ^2/s^2, s^2 being the sample variance. *C. C. Craig* (Ann Arbor, Mich.).

von Schelling, Hermann. Statistische Schätzungen auf kombinatorischer Grundlage. Z. Angew. Math. Mech. 21, 52–58 (1941).

The author's object is to calculate the confidence intervals for the mean of a single finite population and for the difference between two means of arbitrary populations. To solve the first of these problems, the author deduces the formulae for the first three moments of the mean of a sample drawn without replacement and the expectations of the second and third sample moments. The object of calculating the third moment is to show that the distribution of the sample mean is less skew than that of a single observation. The formula suggested for the confidence interval for the mean is the familiar one, $\bar{x}-3s\leqq\xi\leqq\bar{x}+3s$, where s^2 is the estimate of the variance of $\bar{x}$, given by

$$n(n-1)Ns^2=(N-n)\sum(x_i-\bar{x})^2.$$

It is somewhat unexpected that, while introducing the factor $(N-n)/N$ correcting for finiteness of the population, the author ignores the fact that, with small n, the distribution of $(\bar{x}-\xi)/s$ may be far from normal and that consequently the constant factor 3 may be inappropriate. The formulae for moments and expectations and the conclusions drawn from them are also familiar since at least 1923 [P. S. Dwyer, Ann. Math. Statistics 9, 97 (1938)].

The second part of the paper deals with confidence intervals for the difference δ between proportions p_1 and p_2 of black balls in two bags. If samples of n_1 and n_2 are drawn without replacement and the numbers of balls in the bags are N_1 and N_2, the author writes $n_i'=n_i(N_i-1)/(N_i-n_i)$. Denoting by x and y the observed relative frequencies of black balls, the author finds the variance

$$\sigma^2=p_1(1-p_1)/n_1'+p_2(1-p_2)/n_2'$$

of $d=x-y$ and presumes that the ratio $(d-\delta)/\sigma$ varies normally. He then calculates the minimum D' and the maximum D'' of values of δ such that $(d-\delta)^2/\sigma^2=k^2=\text{const.}$, and suggests that the formula $D'\leqq\delta\leqq D''$ gives a confidence interval for δ. The author notices, however, that the limits (D', D'') are somewhat too biased.

The third part of the paper deals with confidence limits for the difference between means of two arbitrary populations. The idea and the calculations are essentially covered by the paper of E. J. G. Pitman [Suppl. J. Roy. Statist. Soc. 4, 119 (1937)]. The difference consists in that the author does not notice the necessity of Pitman's restrictions concerning the populations sampled and in that he does not make any attempt to study the distributions of the statistics used. A remark added in proof quotes the paper by Pitman, but the author does not seem to appreciate fully his results. *J. Neyman* (Berkeley, Calif.).

Bernstein, S. On "fiducial" probabilities of Fisher. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 85–94 (1941). (Russian. English summary) [MF 4507]

This is a record of the author's contribution to a discussion held at a conference on probability, the other participants being Kolmogoroff and Romanowski. The concept of fiducial probability is found to involve inconsistencies similar to those already mentioned in the literature [J. Neyman, Lectures and Conferences on Mathematical Statistics, Washington, D. C., 1938]. *J. Neyman.*

Baker, G. A. Maximum likelihood estimation of the ratio of the components of non-homogeneous populations. Tôhoku Math. J. 47, 304–308 (1940). [MF 4030]

Let the probability distribution function of a variate x be given by $1/(1+k)[f_1(x)+kf_2(x)]$, where $f_1(x)$ and $f_2(x)$ are given distribution functions and k denotes an unknown positive number. The author discusses some difficulties which arise in the application of the method of maximum likelihood to the problem of estimating the unknown parameter k on the basis of a small sample. It is, however, not shown that in the cases considered some other methods of estimation exist which lead to better results. The reviewer should like to remark that the optimum properties of the maximum likelihood estimates have been proved only for large samples under certain assumptions on the probability distribution $f(x)$. *A. Wald* (New York, N. Y.).

Baker, G. A. Test of homogeneity for normal populations. Ann. Math. Statistics 12, 233–236 (1941). [MF 4716]

Let $x_1, x_2, \cdots, x_{n+1}$ be $n+1$ independent observations of the same variable x following the probability law

$$p(x)=(\sigma\sqrt{2\pi})^{-1}[p\exp(-(x-m_1)^2/2\sigma^2) +q\exp(-(x-m_2)^2/2\sigma^2)]$$

with unknown parameters m_1, m_2, σ and $q=1-p$. The author considers the hypothesis H that $m_1=m_2$ and suggests the following test. If the hypothesis is true and $\bar{x}$ and s denote the mean and the S.D. of any n of the $n+1$ observations, while x is the $(n+1)$st, then $x'=(x-\bar{x})/s(n+1)^{\frac{1}{2}}$ follows the "Student's" distribution. A sample of $n+1$ observations determines n variables x'. These are mutually dependent, but for large values of n the correlations are small. The author suggests that the hypothesis H should be tested by applying to the n values of x' the "smooth" test for goodness of fit [J. Neyman, Skand. Aktuarietidskr. **1937**, 149–199], as if they were independent. The paper ends with results of four sampling experiments illustrating the power of the test as applied in four different situations. As it is the relative frequency with which a given test acts in this or that way that is important, one could regret that the number of sampling experiments is only four and not, say, one hundred in each situation studied. *J. Neyman.*

Gordon, Robert D. The estimation of a quotient when the denominator is normally distributed. Ann. Math. Statistics 12, 115–118 (1941). [MF 4010]

Let x and y be two independently distributed random variables with mean values a and b, respectively. The author deals with the problem of obtaining an unbiased estimate of the ratio b/a by means of observations on x and y. A function $\mathfrak{S}(x)$ is given whose expected value is equal to $1/a$ if x is normally distributed and $a\neq 0$. Then $y\mathfrak{S}(x)$ is an unbiased estimate of b/a. By a known theorem, the arithmetic mean of n independent observations on $\mathfrak{S}(x)$ converges stochastically to $1/a$ with $n\rightarrow\infty$, and the author investigates the "speed" of this stochastic convergence.

A. Wald (New York, N. Y.).

Hsu, P. L. Canonical reduction of the general regression problem. Ann. Eugenics 11, 42–46 (1941). [MF 4862]

Suppose $(x_i-\sum_{u=1}^{q}\beta_{iu}w_u)$ $(i=1, 2, \cdots, p)$ are distributed according to a normal multivariate distribution with matrix $\|\alpha_{ij}\|$, the x_i being random variables and the w_u "fixed" variables. The general linear regression hypothesis H which has been studied by Daly [Ann. Math. Statistics 11, 1–32 (1940); cf. these Rev. 1, 347] is that $\beta_{is}=0$ $(i=1, 2, \cdots, p;$ $s=1, 2, \cdots, n_1\leqq q)$, where the hypothesis is to be tested, of course, on the basis of a sample (x_{ir}, w_{ur}) $(i=1, 2, \cdots, p;$ $u=1, 2, \cdots, q;$ $r=1, 2, \cdots, N\geqq p+q)$, where $\|w_{ur}\|$ has rank q. By using a linear transformation Hsu reduces this problem to one of a certain "canonical" form. More specifically, by taking certain linear functions $y_{i\mu}$ and z_{it} of the x_{ir} $(\mu=1, 2, \cdots, q;$ $t=1, 2, \cdots, N-q)$ he reduces the normal multivariate distribution of $(x_{ir}-\sum_{u=1}^{q}\beta_{iu}w_{ur})$ $(r=1, 2, \cdots, N)$ to a product of normal multivariate distributions of the $(y_{i\mu}-\eta_{i\mu})$ $(\mu=1, 2, \cdots, q)$ and z_{it} $(t=1, 2, \cdots, N-q)$, both having the matrix $\|\alpha_{ij}\|$ for each value of μ and t, where the $\eta_{i\mu}$ are homogeneous linear functions of the β_{iu} such that in η_{is} $(i=1, 2, \cdots, p;$ $s=1, 2, \cdots, n_1)$ the coefficients of β_{iu} $(u>n_1)$ are all zero. Thus the hypothesis H reduces to the hypothesis H_0 that $\eta_{is}=0$.

Hsu also discusses briefly the problem of the distribution of generalized canonical correlations which he solved in 1939 from the point of view of the "canonical" distribution of the $y_{i\mu}$ and z_{it}. *S. S. Wilks* (Princeton, N. J.).

Hsu, P. L. On the problem of rank and the limiting distribution of Fisher's test function. Ann. Eugenics 11, 39–41 (1941). [MF 4861]

Assuming that $d_i=(x_i-\sum_{u=1}^{q}\beta_{iu}w_u)$ $(i=1, 2, \cdots, p)$ are distributed according to a normal multivariate law, the author considers the problem of testing the hypothesis H that the rank of the matrix $\|\beta_{ii'}\|$ $(i'=1, 2, \cdots, n_1<q)$ is l; the test to be made on the basis of a sample (x_{ir}, w_{ur}) $(r=1, 2, \cdots, N\geqq p+q;$ $i=1, 2, \cdots, p;$ $u=1, 2, \cdots, q)$, where the rank of $\|w_{ur}\|$ is q. Let

$$\|a_{ij}\|=\mathbf{XMW}_1'(\mathbf{W}_1\mathbf{MW}_1')^{-1}\mathbf{W}_1\mathbf{MX}', \quad i, j=1, 2, \cdots, p,$$

and

$$\|a_{ij}+b_{ij}\|=\mathbf{XMX}',$$

where $\mathbf{X}=\|x_{ir}\|$, $\mathbf{W}_1=\|w_{i'r}\|$ $(i'=1, 2, \cdots, n_1; r=1, 2, \cdots, N)$ and $\mathbf{M}=\mathbf{I}-\mathbf{W}_2'(\mathbf{W}_2\mathbf{W}_2')^{-1}\mathbf{W}_2$, $\mathbf{W}_2=\|w_{i''r}\|$ $(i''=n_1+1, \cdots, q;$ $r=1, 2, \cdots, N)$. The criterion, say Φ, proposed for testing H is the sum of the $l-\min(p, n_1)$ smallest non-vanishing roots of the equation $|a_{ij}-\varphi b_{ij}|=0$. It is shown that the limiting distribution of $N\Phi$ as $N\rightarrow\infty$, and assuming H to be true, is the χ^2-distribution with $(p-l)(n_1-l)$ degrees of freedom. *S. S. Wilks* (Princeton, N. J.).

MacStewart, W. A note on the power of the sign test. Ann. Math. Statistics 12, 236–239 (1941). [MF 4717]

Consider a set of non-zero differences of which x are positive and $N-x$ are negative and assume that the hypothesis H_0 tested is that the expected distribution of x about pN is given by the binomial $(p+q)^N$, where $p=q=\frac{1}{2}$. Let r be the smaller of x and $N-x$. A table is presented whereby we may answer the following two questions: (1) What is the minimum value of N for a given significance level $\epsilon\leqq .05$ that will give the test of H_0 a power $P>\beta$ relative to an alternative hypothesis H_1 corresponding to some specified value $p=p_1$? (2) For this minimum value of N, what is the maximum value of r? The table is given for values of p_1 from .60 to .95 at intervals of .05 and for values of β from .05 to .95 at intervals of .05 and also for $\beta>.99$. The application of the table is illustrated.

W. A. Shewhart (New York, N. Y.).

Wald, Abraham. Asymptotically most powerful tests of statistical hypotheses. Ann. Math. Statistics 12, 1–19 (1941). [MF 4001]

Let $f(x|\theta)$ denote the elementary probability law of a random variable x depending on a single unknown parameter θ, let H denote a simple hypothesis that $\theta=\theta_0$, Ω_1 and Ω_2 sets of admissible hypotheses specifying the values of θ respectively not greater than θ_0 and both $\theta\leqq\theta_0$ and $\theta>\theta_0$. Further, let W_n be a critical region for testing H against Ω using n independent observations $E_n=(x_1, x_2, \cdots, x_n)$ of x and let $P(W_n|\theta)$ be the probability of $E_n\epsilon W_n$ considered as a function of θ. The following two new conceptions are introduced. (1) A sequence $\{W_n\}$, where $n=1, 2, \cdots$, is said to determine an asymptotically most powerful test of H at the level of significance α if $P\{W_n|\theta_0\}=\alpha$ and if any sequence $\{Z_n\}$ of regions for which $P(Z_n|\theta_0)=\alpha$ satisfies the inequality

$$\limsup_{n\rightarrow\infty}\{\underset{\theta}{\text{G. L. B.}}[P(Z_n|\theta)-P(W_n|\theta)]\}\leqq 0.$$

(2) The sequence $\{W_n\}$ determines an asymptotically most powerful unbiased test of H, at the level of significance α, if $P(W_n|\theta)\geqq P(W_n|\theta_0)=\alpha$ and if the conditions $P(Z_n|\theta)\geqq P(Z_n|\theta)=\alpha$ imply

$$\limsup_{n\rightarrow\infty}\{\text{G. L. B.}[P(Z_n|\theta)-P(W_n|\theta)]\}\leqq 0.$$

Let $\hat{\theta}_n$ denote the maximum likelihood estimate of θ. The main results of the paper consist of theorems asserting that,

under fairly general conditions, (a) the test of H against Ω_1 consisting of rejecting H whenever $\hat{\theta}_n$ is too small is an asymptotically most powerful test, (b) the test of H against Ω_2 consisting of rejecting H whenever $\hat{\theta}_n$ differs too much from θ_0 in either direction is an asymptotically most powerful unbiased test. *J. Neyman* (Berkeley, Calif.).

Wald, A. and Wolfowitz, J. Note on confidence limits for continuous distribution functions. Ann. Math. Statistics 12, 118–119 (1941). [MF 4011]

Attention is called to papers by A. Kolmogoroff [Giorn. Ist. Ital. Attuari 4, 83–91 (1933)] and by N. Smirnoff [Rec. Math. [Mat. Sbornik] N.S. 6 (48), 3–26 (1939); cf. these Rev. 1, 246] anticipating some results of the authors [Ann. Math. Statistics 10, 105–118 (1939)] and providing easy methods for constructing confidence limits for a cumulative distribution function known to be continuous.

J. Neyman (Berkeley, Calif.).

Wald, A. and Brookner, R. J. On the distribution of Wilks' statistic for testing the independence of several groups of variates. Ann. Math. Statistics 12, 137–152 (1941). [MF 4707]

By using the Neyman-Pearson likelihood ratio method Wilks [Econometrica 3 (1935)] has derived a criterion for testing the hypothesis H that k specified groups of the variates in a given normal multivariate population are mutually independent. The criterion found was

$$\lambda=|r_{ij}|/(|r_{\alpha_1\beta_1}|\cdot|r_{\alpha_2\beta_2}|\cdot\cdots\cdot|r_{\alpha_k\beta_k}|)^{-1},$$

where $|r_{ij}|$ is the sample determinant of correlation coefficients among all variates and $|r_{\alpha_t\beta_t}|$ is that for the variates in the tth group of variates. The moments of λ were found when the hypothesis H is true. In some of the simpler cases it was shown by solving certain moment equations that the exact distributions of λ are simple elementary functions. By making use of characteristic functions Wald and Brookner provide a method for finding the exact distribution of λ if there is an odd number of variates in at most one group. They also give an expansion of the exact cumulative distribution in an infinite series which is applicable to any grouping. Denoting $-\log\lambda$ by v, their expansion is as follows:

$$P(v>v_0)=\left(\frac{2}{n}\right)^r C_n\left\{\left[1-I\left(\frac{nv_0}{2r^{\frac{1}{2}}},r-1\right)\right]+\frac{\beta_1}{n}\left[1-I\left(\frac{nv_0}{2(r+1)^{\frac{1}{2}}},r\right)\right]+\frac{\beta_2}{n^2}\left[1-I\left(\frac{nv_0}{2(r+2)^{\frac{1}{2}}},r+1\right)\right]+\cdots\right\},$$

where n is the sample size, $4r=(\sum_1^k r_t)^2-\sum_1^k r_t^2$, where r_t is the number of variables in the tth group, the β's are constants depending on the r_t and C_n is a constant depending on n and the r_t; $I(u, p)$ is the incomplete beta function. The authors give tables of values of r, β_1, β_2, β_3, β_4 for all groupings for which $2\leqq k\leqq 6$ and $3\leqq\sum_1^k r_t\leqq 6$, and also tables of $(2/n)^r C_n$ for $n=10, 11, \cdots, 20, 22, \cdots, 30, 35, \cdots, 70, 80, \cdots, 100$, and for the same groupings of variates. *S. S. Wilks* (Princeton, N. J.).

Craig, Cecil C. Note on the distribution of non-central t with an application. Ann. Math. Statistics 12, 224–228 (1941). [MF 4714]

N. L. Johnson and B. L. Welch [Biometrika 31, 362–389 (1940); cf. these Rev. 1, 346] have applied an approximate procedure for tabling the distribution of the so-called non-central t, given by $t=(z+\delta)/\psi$, where z is a normal variate about zero with unit S.D., δ an arbitrary constant and $\psi^2 f$ a variable independent of z following the χ^2 distribution with f degrees of freedom. The author deduces an exact formula for the probability of the non-central t falling within any interval $(-t_0, t_0)$, namely

$$P\{-t_0\leqq t\leqq t_0\}=e^{-\delta^2/2}\sum_{r=0}^{\infty}\frac{(\delta^2/2)^r}{r!}I(r+\tfrac{1}{2}, \tfrac{1}{2}f; t_0^2/(f+t_0^2)).$$

[There is a misprint in the formula (5) giving this probability.] This formula (in its correct form) is used to appreciate the power of the "randomization" test of the hypothesis that two populations have equal means, as suggested by Fisher [Design of Experiments, Oliver and Boyd, Edinburgh, 1937] and developed by E. J. G. Pitman [Suppl. J. Roy. Statist. Soc. 4, 119–130 (1937)].

Assuming that the approximate distribution of the criterion w of the randomization test as found by Pitman is the exact one [denote this assumption by A], the author calculates the power of this test, that is, the probability of the test rejecting the hypothesis tested when this is inexact. In so doing, the author is forced to specify the distributions in the populations sampled and assumes that they are both normal with equal variances [this we shall call assumption B], which makes it possible to apply the formula quoted above. The paper ends with two short illustrative tables, giving the power of the test. Under the assumptions A and B, the problem of the author is identical with that of the power of the t test in its symmetrical form and is covered by a more general problem of the power of the analysis of variance test. The author seems to have overlooked that the treatment of the latter is given by P. C. Tang [Statist. Res. Mem. London 2, 126–157 (1938)]. *J. Neyman.*

Subramanian, S. Compatibility of Fisher's tests for index number formulae. Math. Student 8, 124–127 (1940). [MF 4315]

Wilks, S. S. Determination of sample sizes for setting tolerance limits. Ann. Math. Statistics 12, 91–96 (1941). [MF 4006]

In applied statistics, one of the most important practical problems is the establishment of valid tolerance ranges. This paper breaks new ground by providing a mathematical method for establishing the simplest type of tolerance range under conditions of statistical control and also defines mathematically an important type of tolerance range problem that remains unsolved. A method is given for determining the size of sample required for setting tolerance limits on a statistically controlled variable x having any unknown continuous distribution $f(x)$ and having a previously specified degree of stability. The special case when $f(x)$ is normal is discussed. Illustrative examples are cited. It is significant for the practicing statistician that large samples are required in order to establish tolerance ranges with the degree of stability customarily demanded.

W. A. Shewhart (New York, N. Y.).

Neyman, J. On a statistical problem arising in routine analyses and in sampling inspections of mass production. Ann. Math. Statistics 12, 46–76 (1941). [MF 4003]

The author considers the particular case where the values of Nn random variables $x_{i,j}$, $i=1, 2, \cdots, N; j=1, 2, \cdots, n$,

can be observed and it is known that $x_{i,j}$ is independent of $x_{k,l}$ for $i\neq k$ and

$$p(x_{i,1}, x_{i,2}, \cdots, x_{i,n}) = \left(\frac{1}{\sigma_i(2\pi)^{\frac{1}{2}}}\right)^n \exp\left(-\tfrac{1}{2}\sum_{j=1}^{n}(x_{i,j}-\xi_i)^2/\sigma_i^2\right),$$

with unknown values of ξ_i and $\sigma_i>0$. The hypothesis H to be tested with specifically defined errors of the first and second kind is that the standard deviation of routine analysis remains constant or that $\sigma_1=\sigma_2=\cdots=\sigma_N=\sigma$, where the actual value of σ is unknown and n is assumed to be small. The author's purpose is to select a region from an infinite family of all regions similar to the sample space in respect to the $N+1$ nuisance parameters $\sigma, \xi_1, \xi_2, \cdots, \xi_N$ which he judges most satisfactory for controlling errors of the second kind. After a consideration of the types of choices to be made in choosing a set of hypotheses alternative to H, a specific choice is made and it is shown that there is no uniformly most powerful test for testing H against any and all simple alternatives of the type chosen. By making further limiting assumptions about the hypothesis alternative to H to be tested, a critical region of unbiased type is obtained and its method of use is indicated upon the further assumption that N is large. The limitations of the proposed test of the hypothesis that the error of measurement is constant from the viewpoint of the important practical problem of controlling the assignable causes of such variation, if such causes exist, are not considered and the term accuracy is substituted throughout the paper for what is commonly called precision. *W. A. Shewhart.*

Mosteller, Frederick. Note on an application of runs to quality control charts. Ann. Math. Statistics **12**, 228–232 (1941). [MF 4715]

Let us consider a sample of $2n$ drawn one at a time from a continuous distribution function $f(x)$ and arranged in the order in which they were drawn. Designate a value x_i less than the median of the sample by the symbol a and a value greater by the symbol b. Making use of theory previously developed by A. M. Mood [Ann. Math. Statistics **11**, 367–392 (1940); cf. these Rev. **2**, 228] the author gives a table of the smallest length of runs of a's and b's to be considered significant for .05 and .01 significance levels for samples of size 10, 20, 30, 40 and 50, both for runs on one side of the median and for runs on either side of the median. He also presents a table furnished by P. S. Olmstead and giving the probabilities of getting at least one run of s or more on one side, either side and each side of the median for samples of size 10, 20 and 40. The use of these tables in the control of quality of manufactured product is indicated.
W. A. Shewhart (New York, N. Y.).

Faesi, M. Über die Glättung statistischer Verteilungsreihen. Mitt. Verein. Schweiz. Versich.-Math. **40**, 61–84 (1940). [MF 4391]

The author considers a two-way infinite series $\cdots+A_{-2}+A_{-1}+A_0+A_1+A_2+\cdots$ of non-negative terms, having a sum equal to unity. From this series, a new series is to be obtained by using the moving average over $2k+1$ terms. To this new series also is applied the moving average over $2k+1$ terms; and the process is continued n times. At each stage, the set of terms obtained may be represented by a step-function. The author shows that under certain conditions, and with a scale depending upon $n^{\frac{1}{2}}$, this step function approaches uniformly the Gaussian distribution $h\pi^{-\frac{1}{2}}\exp(-h^2x^2)$, where $h^2=6/[(2k+1)^2-1]$. The conditions imposed imply that the auxiliary series $\sum_{-\infty}^{\infty}A_\lambda s^\lambda$ converges in a ring in the complex plane, in which lies the circle with unit radius. For effecting the proof, the author uses an operator S, which shifts the origin of the subscripts of the A's, and a linear operator G which, applied to an α series and a β series, leads to $G(a\alpha+b\beta)=aG(\alpha)+bG(\beta)$, where $a\geqq 0$, $b\geqq 0$ and $a+b=1$. It might be noted that E. Slutsky [Econometrica **5**, 105–146 (1937)], starting from other hypotheses, indicates that repeated summation of random elements leads toward the Gaussian distribution, and on p. 112 gives a figure showing a close fit to a Gaussian curve obtained by ten successive summations by three.
E. L. Dodd (Austin, Tex.).

Dodd, Edward L. The cyclic effects of linear graduations persisting in the differences of the graduated values. Ann. Math. Statistics **12**, 127–136 (1941). [MF 4706]

It is known that repeated summing or averaging applied to a series of independent observations on a chance variable produces cyclical effects in the graduated values. In this paper the author deals with the question of whether the cyclical effects appearing in the graduated values persist in their successive differences. In particular the case is studied when $k+2$ successive summings by n are applied to the chance data, and then k, or $k+1$, or $k+2$ differencings of the graduated values are performed. The author finds that in the case of $k+2$ differencings the resulting sequence has something of the same chaotic nature as the original data. In particular for $n\geqq 2$ the expected frequency of changes of sign is the same as in the original data. But with only k or $k+1$ differencings a definite cyclicity remains, especially if n is large. *A. Wald* (New York, N. Y.).

Kendall, M. G. The effect of the elimination of trend on oscillations in time-series. J. Roy. Statist. Soc. (N.S.) **104**, 43–52 (1941). [MF 4570]

Gumbel, E. J. The return period of flood flows. Ann. Math. Statistics **12**, 163–190 (1941). [MF 4709]

Applications of the Theory of Probability, Economics

Borel, Émile. Théorie de l'hérédité: définitions et problèmes. C. R. Acad. Sci. Paris **212**, 777–780 (1941). [MF 4932]

Borel, Émile. Sur certains problèmes d'hérédité connexes au problème de la ruine des joueurs. C. R. Acad. Sci. Paris **212**, 821–825 (1941). [MF 5028]

The author sketches a formal treatment of Mendelian theory in which an individual is defined as a set of $2n$ chromosomes, the size of the population increases steadily according to an exponential law, etc. The problem of survival of a special kind of chromosome in the nth generation is closely related to the classical problem of ruin in the binomial case and can be treated in a similar way by means of a generating function. *W. Feller* (Providence, R. I.).

Mittmann, Otfrid. Funktionale Zusammenhänge zwischen Zygotenwahrscheinlichkeiten. Deutsche Math. **5**, 563–570 (1941). [MF 4804]

In the usual treatment of the probable distribution of

genes in a population it is assumed that the population is absolutely homogeneous. The author deduces the functional relationships between the different probabilities under the more general assumption that the probability of the choice of parents depends on their genes, and also takes into consideration a natural selection. *W. Feller.*

Feller, Willy K. Statistical aspects of ESP. J. Parapsychology 4, 271–298 (1940). [MF 5610]

Greenwood, J. A. and Stuart, C. E. A review of Dr. Feller's critique. J. Parapsychology 4, 299–319 (1940). [MF 5611]

Lotka, Alfred J. Sur une équation intégrale de l'analyse démographique et industrielle. Mitt. Verein. Schweiz. Versich.-Math. **40**, 1–16 (1940). [MF 4388]

In view of a criticism raised by Hadwiger [Mitt. Verein. Schweiz. Versich.-Math. **38**, 1–14 (1939); cf. these Rev. **1**, 154] the author gives a new exposition of his method [the most complete exposition of which appeared in Ann. Math. Statistics **10**, 1–25 (1939)]. On special examples it is shown how the method works in practice. *W. Feller.*

Hadwiger, Hugo. Über eine Funktionalgleichung der Bevölkerungstheorie und eine spezielle Klasse analytischer Lösungen. Bl. Versich.-Math. **5**, 181–188 (1941). [MF 4577]

The author again [cf. these Rev. **1**, 154, 349; **2**, 238] considers the integral equation of population analysis in the form

$$(*)\qquad G(t)=\int_0^\infty G(t-\xi)\phi(\xi)d\xi;$$

this time it is supposed that $\phi(\xi)=e^{-\xi}\sum_{k=0}^n A_k\xi^k$, and $\phi(\xi)\geqq 0$ for $\xi\geqq 0$. It is shown that, under some further assumptions, the solution of (*) is given by $G(t)=\Re\sum_{k=0}^n C_k e^{r_k t}$, where r_k are the (simple) roots of the characteristic equation $\sum_{k=0}^n A_k(1+r)^k=(1+r)^{n+1}$ with $\Re(r_k)>-1$. *W. Feller.*

Hadwiger, H. Natürliche Ausscheidefunktionen für Gesamtheiten und die Lösung der Erneuerungsgleichung. Mitt. Verein. Schweiz. Versich.-Math. **40**, 31–39 (1940). [MF 4389]

As on a previous occasion [Skand. Aktuarietidskr. **1940**, 101–113; cf. these Rev. **2**, 238], the author remarks that the integral equation of renewal theory

$$(*)\qquad \phi(t)=f(t)+\int_0^t f(t-\xi)\phi(\xi)d\xi$$

can be solved explicitly if

$$f(t)=a\pi^{-\frac{1}{2}}x^{-\frac{3}{2}}\exp\{ca-\Lambda x-a^2x^{-1}\}.$$

This function is one of the solutions of a functional equation to which the author seems to attach a metaphysical importance. *W. Feller* (Providence, R. I.).

Hadwiger, H. Eine Formel der mathematischen Bevölkerungstheorie. Mitt. Verein. Schweiz. Versich.-Math. **41**, 67–73 (1941). [MF 5099]

The author again gives a treatment of the equation (*) of the preceding review for the case where $f(t)=at^ne^{-ct}$ ($a, n, c>0$; n an integer). [Cf. Hadwiger and Ruchti, Metron **13**, no. 4, 17–26 (1939); these Rev. **1**, 349.]

W. Feller (Providence, R. I.).

Féraud, L. Le renouvellement, quelques problèmes connexes et les équations intégrales du cycle fermé. Mitt. Verein. Schweiz. Versich.-Math. **41**, 81–93 (1941). [MF 5100]

This paper reviews different methods which have been proposed for the solution of the equation (*) of the second preceding review, and different special cases in which they lead to explicit solutions. For example, if $f(t)$ is a finite sum of exponentials, one obtains $\phi(t)$ easily using the Laplace transformation. *W. Feller* (Providence, R. I.).

***Davis, Harold T. The Theory of Econometrics.** Principia Press, Inc., Bloomington, Ind., 1941. xiv+482 pp.

Econometrics is a relatively young science which has been greatly developed in the last few decades. Its subject is a quantitative formulation of economic theories together with the problem of their statistical verifications. Many of the modern findings are scattered over a great number of periodicals and there has been a need for a comprehensive presentation of the theory. The present book, with its extensive bibliographical references, certainly contributes towards that end.

The book is divided into two parts. The first part deals with economic statics and the second with economic dynamics. The topics discussed in economic statics are: the nature of wealth and income; the concept of utility; supply and demand curves; the theory of pure exchange; the theory of monopoly and duopoly; production functions; budgets; the theory of equilibrium and taxation. Among the topics dealt with in the second part the following may be mentioned: the growth of population and industry with a discussion of the logistic curve; the equation of exchange; index numbers from the economic point of view; time series and their correlation including trends, harmonic analysis and serial correlation; dynamic concepts of supply and demand; the dynamics of economic time series with a discussion of the erratic-shock theory; the theory of business cycles and international exchange.

The book was apparently written to be used as a text. This purpose is served also by the large number of problems in each chapter designed as exercises for students. Mathematical tools are extensively used in the book, although not on a very advanced level, so that a knowledge of calculus is sufficient for the understanding of most of the book. Since the book was written as a text, and not as a treatise, the treatment of the topics is somewhat elementary and not as complete or profound as it could have been in a treatise. In several cases the reader is referred to another larger work of the author [The Analysis of Economic Time Series] for a more extensive analysis. Mathematical economists will find interesting the discussions of numerous modern theoretical results and the extensive statistical data with which the theory is confronted. Teachers of mathematical economics will find the book helpful in their class work.

A. Wald (New York, N. Y.).

Hitchcock, Frank L. The distribution of a product from several sources to numerous localities. J. Math. Phys. Mass. Inst. Tech. **20**, 224–230 (1941). [MF 5066]

Assume m factories shipping products to n cities. Let the cost per ton of shipping from ith factory to jth city be a_{ij} and the number of tons so shipped x_{ij}. Then the total cost $y=\sum_{j=1}^n\sum_{i=1}^m a_{ij}x_{ij}$. Let the total product shipped from the ith factory to the n cities be f_i tons, the total product consumed by the jth city be c_j tons. Then $\sum_j x_{ij}=f_i$, $i=1,\cdots,m$,

$\sum a_{ij}x_{ij}=c_j$, $j=1, \cdots, n$. Problem: Find a combination of the $m\cdot n$ quantities x_{ij} which will make y a minimum. The $m+n$ linear equations and the relation $\sum f_i=\sum c_j$ lead to a solution with $m\cdot n-m-n+1$ of the x_{ij} as parameters. Making use of the fact that all a_{ij} and all x_{ij} are non-negative, the solution is given by a simple and interesting geometrical interpretation in the $m\cdot n$ dimensional space of the x_{ij}. Algebraically, this leads to a systematic reduction by which the solution is obtained by a finite number of trials.

A. J. Kempner (Boulder, Colo.).

GEOMETRY

✱Forder, Henry George. The Calculus of Extension. Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1941. xvi+490 pp. $6.75.

From the preface and introduction: "This book had its origin when Professor Neville in 1929 passed on to me some papers left to the Mathematical Association by Professor Genese. These contained lecture notes and examples on Grassmann's methods. Though the notes themselves have been used but little, a large number of examples from this source have been incorporated in the earlier chapters. As I had been interested in this field for years, I thought it might be worth while to extend the work beyond the strictly elementary field covered in Genese's notes, and give a coherent account of Grassmann's methods, with a number of applications sufficient to justify their use. The emphasis on identities is my own, and my aim has been to express geometric theorems as identities, involving not coordinates but the geometric entities themselves which appear in the theorems." "The algebra of vectors created by Grassmann and Hamilton has at last won an established place in physics. Grassmann's methods are of equal use in geometry, but this application is less widely appreciated. It is hoped that this book will redress the balance."

The book is the first modern textbook of Grassmann's calculus of extension. It gives a very lucid account of Grassmann's original methods and of the investigations of other mathematicians who continued his work. Many concrete applications and examples, which, for the first time, are here collected in a book, clarify and extend the theory. The scope of the book is best characterized by enumerating its chapters: I. Plane geometry. II. Geometry in space. III. Applications to projective geometry. IV. Rotors in space, the screw, and the linear complex. V. Differentiation and motion. VI. Projective transformations on the line, plane and space. VII. The general theory. VIII. Application of the general theory to systems of linear equations and determinants. IX. Transformations and square matrices with applications to central quadrics. X. General theory of inner products. XI. Circles. XII. Oriented circles and systems of circles. XIII. The general theory of matrices. XIV. Quadric spreads in spreads of any step. XV. Algebraic products.

E. Helly (Paterson, N. J.).

✱Malengreau, Julien. Précisions sur l'application des fondements de l'arithmétique aux fondements de la géométrie. F. Rouge & Cie, S. A., Lausanne, Switzerland, 1941. 72 pp.

The author calls for a reform in the teaching of elementary geometry in which (i) the space is to be "built up point by point" instead of being regarded as given initially, and (ii) the space is regarded first as reticulated, then as a rational net, then as "homogeneous" and only as a last stage as continuous. The basic figure adopted by the author is the equilateral simplex (which he calls the "ensemble parfait"). The totality of all points is declared to enjoy no property and is hence called "Chaos." *A. A. Bennett.*

Birkhoff, Garrett. Metric foundations of geometry. Proc. Nat. Acad. Sci. U. S. A. 27, 402–406 (1941). [MF 5074]

A proof of the following theorem is sketched (a detailed proof will be forthcoming): a convex complete metric space S, in which sufficiently small segments are unique and any isometry between two subsets σ and σ' of S can be extended to an isometric mapping of S onto itself, is congruent to a Euclidean hyperbolic or spherical space of some finite dimension. This theorem is partly more general, partly more restrictive than a theorem of the reviewer [Amer. J. Math. **43**, 101–111 (1941); these Rev. **2**, 258], who requires finite compactness and strict internal and external convexity (and thus excludes the spherical case), but considers only congruent triples of the form A, B, C and A', B, C instead of arbitrary congruent sets σ and σ'. *H. Busemann.*

Gillam, Basil E. A new set of postulates for euclidean geometry. Revista Ci., Lima **42**, 869–899 (1940). [MF 4348]

This paper presents a purely metric foundation of three dimensional Euclidean geometry in terms of a single class of undefined elements (points) and one primitive relation (distance). A set of points forms a semimetric space S provided there is attached to each pair p, q a non-negative real number pq so that $pq=qp$ and $pq=0$ if and only if $p=q$. The space S is convex if corresponding to each pair of distinct points p, r there is a point q different from p and r with $pq+qr=pr$, and externally convex if p, r imply the existence of s with $pr+rs=ps$, $r\neq s$. Denoting by $D(p_1, p_2, \cdots, p_k)$ the bordered symmetric determinant $|(p_ip_j)^2|$ $(i, j=0, 1, \cdots, k)$, $p_0p_0=0$, $p_0p_i=1$ $(i=1, 2, \cdots, k)$, of k points of S, the author proves that a semimetric space S is logically equivalent to three dimensional Euclidean space E_3 provided the following postulates are satisfied. (I) S is complete, convex and externally convex. (II) For every triple of points p, q, r, the determinant $D(p, q, r)\leqq 0$. (III) If $D(p, q, r)$ vanishes, then $D(p, q, r, s)=0$ for every point s of S. (IV) There exists at least one quadruple of points with non-vanishing determinant D. (V) The determinant D of each five points of S vanishes. The result is obtained by showing that all of Hilbert's axioms for Euclidean three space follow from these postulates. The case $n=3$ of a theorem by Blumenthal weakening the hypotheses of a theorem of Wilson [Amer. J. Math. **57**, 505–517 (1935)] proves that S (with the above postulates satisfied) is congruent with E_3. The paper under review is, however, quite independent (especially as regards method) of these earlier results, and hence furnishes a new derivation of them. Noteworthy features are the author's proofs of (1) the axiom of Pasch, (2) the parallel postulate and (3) the axiom of closure (if two planes have a point in common they have a line in common). Since the technique employed throughout the paper is algebraic, there is developed an analytic geometry without coordinates.

The paper suffers from poor typography. Omission of radical sign (p. 887), deletion of $>$ from $\geqq$ (p. 897) and suppression of $\subset$ (pp. 873, 882), loss of the factor $D(p, q, r)$

from the right hand side of equation on page 899, together with numerous less serious printer's errors, detract from the easy readability of this interesting paper.

L. M. Blumenthal (Columbia, Mo.).

Steck, Max. Zur Axiomatik der reellen projektiven Geometrie III: Beweise des Fano-Axioms F_1 im Rahmen der synthetischen Geometrie. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1939**, 269–276 (1939). [MF 4572]

Liebmann's system of axioms for real projective geometry [Synthetische Geometrie, 1934] contains Fano's axiom F_1 which asserts that the diagonal points of a complete quadrangle are not collinear. The author shows that this axiom is provable in the system and hence may be suppressed. The almost trivial proof given is based principally upon the weak E.P. Axiom, which postulates, corresponding to each non-degenerate conic K_2, at least one point which is the vertex of a pencil of lines intersecting K_2 in two distinct points, and a result of Baldus [S.-B. Bayer. Akad. Wiss. **1932**, 149–191] according to which it suffices to establish Fano's assertion for a single complete quadrangle. Thus the author has removed both Fano axioms F_1, F_2 from Liebmann's system [concerning F_2 see these Rev. **2**, 135, review of a paper by Steck].

L. M. Blumenthal.

Ancochea, G. On the fundamental theorem of projective geometry. Revista Mat. Hisp.-Amer. (4) **1**, 37–42 (1941). (Spanish) [MF 4725]

The author considers projective geometry over an abstract field K. Defining projectivity (in the sense of Poncelet) as a biuniform correspondence obtained by means of projections and sections, he is concerned with what form the fundamental theorem of real projective geometry (valid whenever K is commutative) takes in the case of a noncommutative fundamental field. It is proved that, in any projectivity between two point ranges on the same line r which has three double points, corresponding points x', x of the ranges are given by the interior automorphism $x'=a^{-1}xa$, where a is a fixed point of r and the three double points are selected as the reference points on r. Conversely, it is shown that corresponding to any given automorphism of K there exists a projectivity between two point ranges with the same base which has three double points and in which corresponding points are related by the given automorphism. It follows that for a projectivity between two point ranges on r, with three double points, to reduce to the identity it is necessary and sufficient for a to belong to the centrum of K. Hence the fundamental theorem for commutative fields K is an immediate corollary.

L. M. Blumenthal.

Melchior, E. Über Vielseite der projektiven Ebene. Deutsche Math. **5**, 461–475 (1941). [MF 4798]

A polygram (Vielseit) of order n is defined as the figure formed by any n lines in the real projective plane, with their points of intersection and the polygonal regions into which they divide the plane. The points where only two lines meet are said to be simple. It is proved that, when n is even, the regions can be colored alternately black and white; and that, when n is odd, the segments can be directed in such a way as to give a definite orientation to each region. (In the Euclidean or spherical plane, a polygram admits both these processes for any n.) The rest of the paper is concerned with the special case when all the regions are triangular, and therefore just three of the points are simple. A subset of the points of such a polygram is called a system of kernel-points if each triangle has just one vertex belonging to this subset. It is shown that a polygram has one or three systems of kernel-points according as n is even or odd. In the former case, whenever a line is removed or inserted in such a way that the regions remain all triangular, the kernel-points remain kernel-points. Thus repeated removal or insertion of lines gives rise to a "chain" of polygrams, having a common set of kernel-points. Since a polygram with n odd has two other systems of kernel-points, several chains may be linked into a "chain-system." The known polygrams fall into six chain-systems. [But the diagrams which profess to portray these known polygrams are strangely unintelligible.]

H. S. M. Coxeter (Toronto, Ont.).

Fenchel, W. On the projective geometric foundations of the non-Euclidean trigonometry. Mat. Tidsskr. B. **1941**, 18–30 (1941). (Danish) [MF 5226]

The paper is concerned with the Cayley-Klein projective interpretation of the trigonometric formulae for the right triangle, the trirectangular quadrilateral and the quinquerectangular pentagon in complex non-Euclidean geometry. It is shown that the totality of these formulae is equivalent to the simple relations found by Möbius between the five cross ratios connected with an arbitrary pentagon in the complex projective plane. The two real cases, elliptic and hyperbolic geometry, are discussed in more detail.

B. Jessen (Copenhagen).

Riabouchinsky, Dimitri. Quelques considérations sur les géométries non euclidiennes. C. R. Acad. Sci. Paris **212**, 141–144 (1941). [MF 4894]

Riabouchinsky, Dimitri. Les trigonométries des espaces à n dimensions. C. R. Acad. Sci. Paris **212**, 208–212 (1941). [MF 4895]

In the first paper the author investigates a trigonometry on the surface $x^2+y^2+\epsilon z^2=1$, the angles in the bundle with vertex $x=y=z=0$ being defined with respect to the absolute quadric $x^2+y^2+\epsilon z^2=0$. The system of formulas of this trigonometry is derived from a few identities concerning the elements and cofactors of a determinant of order three. If ϵ is negative, a representation of the plane Lobachevski-geometry by the trigonometry on a hyperboloid of revolution is obtained. An example concerning this representation is discussed in the second paper. Then it is shown how to modify the fundamental identities in order to obtain the corresponding trigonometry in a space of $n>3$ dimensions. In the case $n=4$ these identities are given explicitly.

E. Helly (Paterson, N. J.).

Bukrejew, B. I. Aus dem Gebiete der hyperbolischen Geometrie. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **18**, 57–69 (1940). (Ukrainian. German summary) [MF 4761]

Der Aufsatz zerfällt in 5 Paragraphen: (1) Abstand zweier Punkte und Lobatschewskijsche Formel, (2) Spitzeck von Lambert, (3) Abstandskurven, (4) Grenzkreise, (5) Abbildung auf Beltramischen Kreis. Die Untersuchung gründet sich auf konforme Abbildung der pseudosphärischen Fläche auf die Poincarésche Halbebene, ohne sich projektiver Betrachtungen zu bedienen.

Author's summary.

Algebraic Geometry

Claeys, A. Sur deux cubiques planes. Mathesis **54**, 166–177 (1940). [MF 5160]

Lorent, H. Sur le quartique de Gutschoven. Mathesis **54**, 118–119 (1940). [MF 5154]

Châtelet, François. Courbes réduites dans les classes de courbes de genre 1. C. R. Acad. Sci. Paris **212**, 320–322 (1941). [MF 4902]

In earlier papers [C. R. Acad. Sci. Paris **206**, 1532–1533 (1938); **208**, 487–489 (1939)] the author discusses, by group theory considerations, the question of transforming an algebraic curve, with coefficients in the rational field R, into a cubic in R by means of a birational transformation in R. In the present paper a similar question is considered for any algebraic field k. A series of curves W_n are constructed through definite points c_i such that the Galois group of this extension is multiply isomorphic to the direct product of two cyclic groups. *V. Snyder* (Ithaca, N. Y.).

Chabauty, Claude. Sur les points rationnels des courbes algébriques de genre supérieur à l'unité. C. R. Acad. Sci. Paris **212**, 882–885 (1941). [MF 5035]

Let $f(x_0, x_1, x_2)=0$ be an algebraic curve of genus $g>0$ whose coefficients generate a finite algebraic number field K_0. The author discusses the solutions of $f=0$ by numbers in a finite extension K/K_0 and states the following theorem. The equation $f=0$ has a finite number of solutions in K if its reduced rank r with respect to K is less than g. The number r is defined as $d-1$, where d is the dimension of the additive vector group of the rational (with respect to K) g-tuples of points on $f=0$ in the g-dimensional affine space over an algebraically closed p-adic field H_p. The correspondence between the g-tuples on $f=0$ with coordinates in H_p and above affine space is set up by a generalization of Jacobi's inversion theorem. For this purpose the author proposes to use a local uniformization of $f=0$ which is p-adically convergent. On substituting such a uniformization in the differentials of first kind on the curve it is possible to define local integrals of the first kind. Finally, Mordell's conjecture that $f=0$ contains a finite number of rational points is rephrased. Some of the important details of proof are not indicated in the author's abstract.

O. F. G. Schilling (Chicago, Ill.).

Purcell, Edwin J. Space Cremona transformations of order $m+n-1$. Bull. Amer. Math. Soc. **47**, 242–246 (1941). [MF 4167]

The author defines a space Cremona transformation of order $m+n-1$ (m, n any integers) as follows: Two rational twisted curves C_n and C_m' of orders n and m have, respectively, $n-1$ and $m-1$ points in common with each of two skew lines d and d'. Each point P of space determines two lines, one intersecting C_n in α and d in β, the other intersecting C_m' in γ and d' in δ. The image of P is the intersection P' of the two coplanar lines $\alpha\delta$ and $\beta\gamma$. The equations and characteristics of this transformation are derived. Two special cases are noteworthy. If $C_n \equiv C_m'$, this transformation becomes the involution treated in a recent paper [Bull. Amer. Math. Soc. **46**, 339–344 (1940); these Rev. **1**, 268]. If $d \equiv d'$, the resulting involution is similar to one defined in a different way by Montesano [Ist. Lombardo, Rend. (2) **21**, 688–690 (1888)]. *T. R. Hollcroft* (Aurora, N. Y.).

Purcell, Edwin J. Cremona involutions determined by two line congruences. Bull. Amer. Math. Soc. **47**, 596–601 (1941). [MF 5051]

The Cremona involutions treated are interesting special cases of a previously studied correspondence formed by two line congruences, each consisting of the lines intersecting a space curve C_i of order n_i and its (n_i-1)-secant d_i, $i=1, 2$ [Bull. Amer. Math. Soc. **46**, 339–344 (1940); these Rev. **1**, 268]. By choosing $n_2=2$ in the above correspondence, an involution I of order $2n_1+3$ results. Its equations and fundamental system are obtained. Three special cases of I are then treated: (1) the secant d_1 is in the plane of C_2; (2) the curves C_1, C_2 and the secants d_1, d_2 all lie on a quadric; and (3) $n_1=3$. *T. R. Hollcroft* (Aurora, N. Y.).

Todd, J. A. Birational transformations with a fundamental surface. Proc. London Math. Soc. (2) **47**, 81–100 (1941). [MF 4702]

Previously [Proc. Edinburgh Math. Soc. (2) **5**, 117–124 (1938); Proc. Cambridge Philos. Soc. **34**, 144–155 (1938)] the author has investigated the behavior of the canonical systems of an algebraic variety under birational transformations with isolated basis points or basis curves. In the present paper, the analogous problem for birational transformations with a basis non-singular surface is solved. The characteristics of the birational transformation with a basis non-singular surface are derived. This transformation is first applied to the canonical series on a variety of four dimensions. The findings are then extended, with proof, to d-dimensional varieties. Finally, the above results are empirically generalized to include birational transformations with a basis non-singular k-dimensional variety. The validity of this extension depends on the assumption of a criterion of equivalence whose proof is not yet available.

T. R. Hollcroft (Aurora, N. Y.).

Kollros, Louis. Une propriété des variétés du second ordre. Comment. Math. Helv. **13**, 108–118 (1940).

The locus of the intersection of orthogonal tangents to a conic is a circle called the orthoptic circle of the conic. Similarly there exists an orthoptic hypersphere of a hyperquadric in a space of $n \geqq 2$ dimensions. A theorem of Steiner states that if, at a point P of a conic, one constructs on the exterior normal of a conic a circle tangent to the conic and with a diameter equal in length to the radius of curvature of the conic at P, this circle is orthogonal to the orthoptic circle. And conversely, if one constructs a circle c' tangent to a conic at one of its points P orthogonal to the orthoptic circle c of the conic, the diameter of c' is equal to the curvature of the conic at P. This paper extends the theorem to say: the diameter of the hypersphere tangent to a hyperquadric Q at one of its points P and orthogonal to the orthoptic sphere of Q is equal to the sum of the $(n-1)$ radii of principal curvature of the hyperquadric at P.

V. G. Grove (East Lansing, Mich.).

Ciani, Edgardo. Sopra le superficie cubiche dotate di infiniti punti di Eckardt. Period. Mat. (4) **20**, 240–245 (1940). [MF 5376]

An Eckardt point of a cubic surface is a point at which the tangent plane meets the surface in three concurrent lines [E. Eckardt, Math. Ann. **10**, 227–234 (1875)]. A general cubic surface has no such point, but for particular surfaces a finite number up to 10 may exist, for which the Hessian of the given surface is irreducible. The author proves that

a cubic surface having two biplanar points has an infinite number of Eckardt points forming the line joining them.
V. Snyder (Ithaca, N. Y.).

Godeaux, Lucien. Sur les points unis symétriques des involutions cycliques appartenant à une surface algébrique. Univ. Nac. Tucumán. Revista A. 1, 283–291 (1940). [MF 4067]

In former papers [cf. these Rev. 2, 14 and 138] the author has treated the self-corresponding points of cyclic involutions on an algebraic surface. Such an involution is generated by a birational transformation of the surface into itself. The present paper completes the results for symmetric self-corresponding points. An application is made to involutions of order seven and genus one with three symmetric self-corresponding points on a surface of order n and genus one.
T. R. Hollcroft (Aurora, N. Y.).

Edge, W. L. Notes on a net of quadric surfaces. IV. Combinantal covariants of low order. Proc. London Math. Soc. (2) 47, 123–141 (1941). [MF 4704]

In a series of earlier papers [Proc. London Math. Soc. (2) 43, 302–315 (1937); (2) 44, 466–480 (1938); J. London Math. Soc. 12, 276–280 (1937)] the author discussed a number of combinantal covariants of a net of quadric surfaces. All those mentioned by earlier writers were of order at least 8. The present paper obtains, largely by geometric methods, various new covariants of orders four and six. The covariant surface F_4 is the dual of Gundelfinger's contravariant φ_4. It has the 28 lines joining pairs of base points for bitangents. No claim is made that the new list of covariants is complete. At the end of the paper a particular case is discussed in which the base points are the vertices of two tetrahedra, each inscribed in the other, thus extending a previous paper on the same particular case [Proc. London Math. Soc. (2) 41, 337–360 (1936)]. In this case the Jacobian curve of the net of quadric surfaces consists of four skew lines and their two transversals. *V. Snyder.*

Togliatti, Eugenio G. Una notevole superficie di 5° ordine con soli punti doppi isolati. Vierteljschr. Naturforsch. Ges. Zürich 85 Beiblatt (Festschrift Rudolf Fueter), 127–132 (1940). [MF 4410]

The author defines the surface F_5 in S_3 as follows: Let φ be a cubic primal in S_5 and r a line on φ. The apparent contour of φ from r on an S_3 is a surface F_5 of order 5 in S_3. It is shown that, if φ has no other singularities than α distinct nodes, the surface F_5 has $\alpha+16$ nodes of which α are projections from r of the nodes of φ. The 16 nodes of F_5 are on a quadric cone of the third species having a given plane through r as vertex and tangent to F_5 along a quintic curve. It is stated that since the author has previously found the maximum number of nodes of a cubic primal in S_5 to be 15, the surface F_5 may have as many as 31 nodes. The author states further that this investigation would indicate that the maximum number of nodes of a quintic surface in S_3 is 31.
T. R. Hollcroft (Aurora, N. Y.).

Archbold, J. W. Multiple intersections on an algebraic V_4. Proc. London Math. Soc. (2) 47, 101–122 (1941). [MF 4703]

In 1937, J. A. Todd [Proc. Cambridge Philos. Soc. 33, 425–437 (1937)] investigated the virtual characteristics of the intersections of systems of V_3's on a V_4 (V_i an algebraic variety of dimension i) when these systems have simple basis curves or surfaces. In the present paper, the above results are extended to systems of V_3's which have multiple basis curves or surfaces. The methods used include the degeneration method devised by B. Segre for similar systems on a V_3. Systems of V_3's on a V_4 are treated which have a basis multiple curve (1) that lies on a given surface, (2) that does not lie on a given surface, and which have a basis multiple surface (1) without improper nodes and (2) with improper nodes. *T. R. Hollcroft* (Aurora, N. Y.).

Muhly, H. T. A remark on normal varieties. Ann. of Math. (2) 42, 921–925 (1941). [MF 5522]

An irreducible algebraic variety is said to be normal (in the arithmetic sense) in the projective space if the ring of the homogeneous coordinates of its general point is integrally closed in its quotient field. This definition is due to the reviewer. It has been proved that the system of hyperplane sections of a normal variety is complete (in other words, a normal variety is also normal in the geometric sense, in which this term was used by the Italian geometers). In the present note it is shown in a very simple fashion that a necessary and sufficient condition that a variety be normal in the arithmetic sense is that the system cut out on the variety by the hypersurfaces of any given order be complete.
O. Zariski (Baltimore, Md.).

Heegaard, Poul. Beiträge zur Topologie der algebraischen Flächen. IV. Die Umgebung von Origo im Imaginären auf der Fläche $y=x^3+z^3-3xz$. Avh. Norske Vid. Akad. Oslo. I. 1940, no. 5, 9 pp. (1941). [MF 4663]

Starting from his Copenhagen dissertation of 1898, later published in French [Bull. Soc. Math. France 44, 161–242 (1916)], the author considered an algebraic surface $f(x, y, z)=0$ as an n-sheeted space constructed on the complex plane (x, y), which is projected from $(x_1, x_2, y_1, 0)$ with ordinates y_2. In a later paper [Ninth Scand. Math. Congress, 1939, pp. 15–22] this process is applied to the hypersphere and interpreted topologically on the known Heegaard diagram. Then [Avh. Norske Vid. Akad. Oslo. I. 1939, no. 1, 12 pp. (1939)] the surface $y^2+z^2=2x$ is similarly treated. In the present paper the surface $y=x^3+z^3-3xz$ is considered, with particular emphasis on the map of the neighborhood of the cusp at the origin. The lines joining the branch-points describe a Möbius band, having three half turns instead of one, as in the case of the paraboloid. The region is mapped on an elementary manifold, bounded by a space homeomorphic with a hypersphere. *V. Snyder.*

Differential Geometry

Haupt, Otto. Linear-ordnungssinguläre Punkte ebener und räumlicher Bogen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1939, 253–263 (1939). [MF 4571]

The (linear) order of a (Jordan) arc in Euclidean n-space is the maximum number of its points of intersection with a linear $(n-1)$-space. The order of a point P on $\mathfrak{B}$ is the minimum of the orders of the partial arcs of $\mathfrak{B}$ that contain P in their interior. [These orders need not exist, of course.] Let $n=2$. Then the set $\mathfrak{S}$ of points of order not less than 3 on $\mathfrak{B}$ is closed and nowhere dense. The author outlines the construction of arcs $\mathfrak{B}$ for which $\mathfrak{S}$ is perfect and has an arbitrarily prescribed Lebesgue measure on $\mathfrak{B}$ and for which every point of order not less than 3 has the same order k as $\mathfrak{B}$ itself ($k=4, 5, 6, \cdots$). [An arc of order 3 is the sum

of a finite number of convex arcs and has, therefore, only a finite number of points of order 3.] $\mathfrak{B}$ may have a tangent everywhere which is either continuous in every point or discontinuous in the points of $\mathfrak{S}$ [cf. Haupt, Jber. Deutsch. Math. Verein. **50**, 256–269 (1940); these Rev. **2**, 297]. For $n=3$, similar constructions are sketched for spheric curves.
P. Scherk (Bloomington, Ind.).

Kasner, Edward. Lineal element transformations which preserve the isothermal character. Proc. Nat. Acad. Sci. U. S. A. **27**, 406–409 (1941). [MF 5075]

The author studies those transformations of lineal elements of the plane which preserve isothermal families of curves. Such transformations are the product of a conformal contact transformation and a non-contact factor which leaves all the points fixed. Several properties of certain specialized types of possible non-contact factors are investigated. *P. Franklin* (Cambridge, Mass.).

De Cicco, John. Lineal element transformations which preserve the dual-isothermal character. Proc. Nat. Acad. Sci. U. S. A. **27**, 409–412 (1941). [MF 5076]

The author studies those transformations of lineal elements of the plane which preserve dual-isothermal families of curves. These are families equilongly equivalent to the points of a straight line. A canonical form for all such transformations is found in terms of Hessian coordinates. The contact part of such transformations is more general than the equilong group. *P. Franklin* (Cambridge, Mass.).

Garnier, René. Extension de la formule de Savary au mouvement le plus général d'un solide. Ann. École Norm. (3) **57**, 113–200 (1940). [MF 4634]

When a plane P glides on a plane P_1, the envelope of the successive positions of a curve C of P is a curve C_1 of P_1, the conjugate curve or profile of C. The formula of Savary permits the computation of the radius of curvature R_1 of C_1, in its point of contact with C, in terms of the corresponding radius R of C. The extension of Savary's formula to three-space presents two different problems: (1) there may be a curve, or a class of curves, C of the moving system, whose successive positions have an envelope C_1. Let M be a point of contact between C and C_1. Find the axis of curvature of C_1 in M, if that of C is known. (2) Let S be the envelope of the successive positions of a moving surface S_1, and M a point of contact of both surfaces. Find the elements of curvature of S_1 in M if the corresponding values with respect to the surface S are known. The author of this paper refers to two publications of G. Koenigs, who studied both problems. In the present paper, the author starts with the fact that the relation between the conjugate surfaces S and S_1 is given by a contact-transformation. Let the second derivatives with respect to S be denoted by r, s, t and let $1:r:s:t:rt-s=X:Y:Z:T:U$. It is a known theorem of contact-transformations that the corresponding quantities X_1, Y_1, Z_1, T_1, U_1 may be obtained by a linear transformation of the quantities $X, Y, \cdots, U$. The author develops two methods, which enable one to find the explicit expressions of this linear transformation, and so he obtains formulas which he thinks to be the natural generalization of Savary's formula. He explains how the results of Koenigs and others can be derived from his fundamental formulas. The same contact-transformation applies also to the problem (1) concerning the relation between the curvature of a curve and its conjugate. A considerable part of the paper is devoted to the study of the motion of ruled surfaces whose envelopes are also ruled surfaces. Besides a few other results, this problem is solved: given a motion in three-space, find all couples of ruled surfaces S and S_1 which, in every moment, have contact along a generatrix. In the last chapter it is shown how, from the general theory, statements about the rolling of a surface S on another surface S_1 can be derived. *E. Helly* (Paterson, N. J.).

Bell, P. O. A characterization of the group of homographic transformations. Bull. Amer. Math. Soc. **47**, 488–493 (1941). [MF 4543]

Let $C, \bar{C}$ be curves corresponding in a conformal representation $w=w(z)$ of the points $z=x+iy$ of a region R of the z-plane on the points $w=u+iv$ of the region $\bar{R}$ of the w-plane. Let $\gamma, \bar{\gamma}, s, \bar{s}$ denote the curvatures and arc lengths of $C, \bar{C}$ at z, w, respectively. Let $ds/d\bar{s}=\lambda(x, y)$. The most general directly conformal transformation carrying circles into circles (including straight lines) is a homographic transformation. It is proved that a necessary and sufficient condition that a conformal transformation be a homographic transformation is that the function $\lambda(x, y)$ satisfy the following identities: $\lambda_{xx}-\lambda_{yy}=0$, $\lambda_{xy}=0$. Under these conditions, at a general point w of $\bar{C}$, $d\bar{\gamma}/d\bar{s}$ is independent of the direction of the curve C at the corresponding point z. These conditions also imply that the group of homographic transformations consists of all of the conformal transformations under which the differential form $d\gamma ds$ is absolutely invariant. The paper also gives in geometrical language the significance of the invariance of the Schwartzian derivative under a homographic transformation and characterizes a general homographic transformation by its unique association with two families of concentric circles.
V. G. Grove (East Lansing, Mich.).

Bell, P. O. The R_λ-correspondent of the tangent to an arbitrary curve of a non-ruled surface. Bull. Amer. Math. Soc. **47**, 509–511 (1941). [MF 4548]

In a previous paper [Trans. Amer. Math. Soc. **46**, 389–409 (1939); cf. these Rev. **1**, 85] the author defines the R_λ-correspondent of the tangent at a point P to a general curve C_λ on a surface S. The paper under review gives the following construction for the R_λ-correspondent. Let Q be a point on C_λ distinct from P; let U and V be the points of intersection of the asymptotic curves passing through P, and let W be the point of intersection of the tangent plane to S at P with the line UV. As Q moves along C_λ toward P, W describes a curve C_w and, if C is not a curve of Segre, nor is tangent to a curve of Segre at P, the limit of W is P. The tangent to C_w at P is the R_λ-correspondent of the tangent to C_λ at P. A proof is also given of the theorem: A curve C_λ is a curve of Segre if and only if for a general point P of C_λ the limit of W as Q tends toward P along C_λ is a point W_0 distinct from P. The point W_0 is the intersection of the directrix of Wilczynski of the first kind with the tangent at P to the corresponding curve $C_{-\lambda}$ of Darboux.
V. G. Grove (East Lansing, Mich.).

Fubini, Guido. On the asymptotic lines of a ruled surface. Bull. Amer. Math. Soc. **47**, 448–451 (1941). [MF 4536]

The author finds all ruled surfaces whose curved asymptotics are projective to each other and proves that each one of these asymptotic curves belongs to a linear complex. Moreover the points on the curved asymptotics corresponding in the projectivity lie on one and the same ruling of the surface. The proof of the converse theorem, stating that, if every curved asymptotic of a ruled surface belongs to a

linear complex, these asymptotics are projectively related, appears in the treatise Geometria Proiettiva Differenziale [Zanichelli, Bologna, 1931] by Fubini and Čech, pp. 112–116 and 266–288. *V. G. Grove* (East Lansing, Mich.).

Bouligand, Georges. Sur les asymptotiques des surfaces réglées. C. R. Acad. Sci. Paris **212**, 415–417 (1941). [MF 4911]

A discussion is made of the curved asymptotics on the ruled $x=az+p$, $y=bz+q$, a, b, p, q being functions of a parameter t. The existence of the first derivatives of a, b, p, q is assumed but not that of the second derivatives and $a'q'-b'p'>h^2>0$. This is accomplished by considering a one parameter family of surfaces $x=Az+P$, $y=Bz+Q$, A, B, P, Q being properly restricted functions of t and ϵ, and $A-a$, $B-b$, $P-p$, $Q-q$, $A'-a'$, $B'-b'$, $P'-p'$, $Q'-q'$ tending uniformly to zero with ϵ. *V. G. Grove.*

Bouligand, Georges. Familles de courbes sur certaines surfaces. C. R. Acad. Sci. Paris **212**, 634–636 (1941). [MF 4923]

A discussion is made of certain families of curves on surfaces $y=y(x, u)$, $z=z(x, u)$ for which the functions $y(x, u)$, $z(x, u)$ possess first derivatives only, or possess second derivatives only, and for which y_x and z_x exist but y_u, z_u do not. The methods used are similar to those of the paper reviewed above. *V. G. Grove* (East Lansing, Mich.).

Rosenfeld, B. Théorie des congruences et des complexes de droites dans un espace elliptique. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 105–126 (1941). (Russian. French summary) [MF 4510]

Following a method of A. Norden, developed in a course on the geometry of lines in 1938–1939, a metric is introduced into the manifold of all straight lines r of elliptic three space R_3 by defining the distance ω of two lines by means of the formula $\cos\omega=\cos\omega_1\cos\omega_2$, where ω_1, ω_2 are the extreme distances ω_1, ω_2 of the lines. This metrical manifold P_4 is isometric with a four dimensional quadric in elliptic five space R_5. By a study of the differential properties of this quadric formulas are derived for the behavior of P_4 in the vicinity of a line r in a congruence, a complex and in P_4 as a whole. Special cases investigated are isotropic, parabolic, isometric, normal congruences, the normals to a surface of zero Gaussian curvature and some others.
D. J. Struik (Cambridge, Mass.).

Dwinger, Ph. Der Satz von Bonnet für geradlinige Flächen im elliptischen Raum. Nieuw Arch. Wiskde (2) **20**, 288–290 (1940). [MF 5214]

Llensa, Georges. Sur les systèmes triples orthogonaux doublement L.-D. C. R. Acad. Sci. Paris **212**, 524–526 (1941). [MF 4918]

Unter den einparametrigen Flächenscharen $u(x, y, z)=C$, die sich zu einem dreifach orthogonalen Flächensystem ergänzen lassen, ist eine spezielle Klasse, die hier L.-D. (Lamé-Darboux) genannt wird, von Darboux angegeben worden. Es wird nun gezeigt: Zu einem vorgegebenen Krümmungsstreifen lässt sich immer ein dreifach orthogonales Flächensystem so konstruieren, dass zwei seiner einparametrigen Flächenscharen zur Klasse L.-D. gehören, und dass eine Fläche einer dieser beiden Scharen den vorgegebenen Krümmungsstreifen enthält. *H. Samelson.*

Cartan, Élie. Sur les surfaces admettant une seconde forme fondamentale donnée. C. R. Acad. Sci. Paris **212**, 825–828 (1941). [MF 5029]

The discussion is descriptive and geometrical in character.
T. Y. Thomas (Los Angeles, Calif.).

Touganoff, N. Sur les lignes situées sur une surface dont la torsion géodésique, la courbure normale et la courbure géodésique sont liées par une relation linéaire à coefficients constants. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 383–385 (1941). [MF 4375]

Alt, Wilhelm. Über die Minimalflächen vom Liouvilleschen Typus. Deutsche Math. **5**, 513–521 (1941). [MF 4801]

A surface is of Liouville type if its line element may be given by

$$ds^2=[\phi(\xi)+\psi(\eta)]\cdot(d\xi^2+d\eta^2);$$

the curves ξ, $\eta=$const. constitute the Liouville net. A minimal surface is first considered as a translation surface with minimal curves as parametric curves; imposing the condition that it is of Liouville type one arrives at a line element depending on a polynomial which involves exponential functions. On the other hand, starting with the Weierstrass integral representation, imposing the above condition and transforming to the same parameters, one arrives at elliptic functions. The conclusion is that these should be degenerate, and this leads to the result: A necessary and sufficient condition for a minimal surface to be of Liouville type is that it is an isometric map of a surface of revolution; except when the surface is a plane there is only one Liouville net and it is the image of the net of parallel circles and meridians of the surface of revolution. *G. Y. Rainich.*

Blank, J. Surfaces minima comme surfaces de translation. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 45–61 (1940). (Russian. French summary) [MF 4731]

A translation surface with respect to a plane ω is the transform of a surface of translation under the projective transformation which transforms the plane at infinity into ω. In this paper the question of minimal translation surfaces is considered and it is shown that, if ω is not minimal, the translation surface is imaginary and has the equation $x^2+y^2+z^2-2\beta x^3(x+iy)=0$. If the plane ω is minimal, the translation surface is of the sixth order with the equation $x^2+y^2+z^2-5\sigma^2(x+iy)^6-5\sigma z(x+iy)^3-2s(x+iy)^4=0$.
M. S. Knebelman (Pullman, Wash.).

Blank, J. Zum Engelschen Problem betreffend Translationsflächen. II. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 99–107 (1940). (Russian. German summary) [MF 4751]

This paper deals with Engel's problem of determining all translation surfaces with respect to two planes ω and ω' [cf. above review]. The problem is solved under the restriction that both families of translation curves are plane and one of them belongs to a pencil of planes whose axis is the line of intersection of ω and ω'. There are four possibilities depending on the vanishing or non-vanishing of two invariants, the equations of the corresponding surfaces being (1) $z=x\cdot\operatorname{sh}1/x\cdot e^y$, (2) $z=y^2+y/3x^2-1/45x^4$, (3) $z=\rho xy/2+(\gamma/4)(3y^2+1/x^2)$, (4) $z=x\cdot e^y(\alpha\operatorname{ch}(\alpha^{1/2}/2x)+\delta\operatorname{sh}(\alpha^{1/2}/2x))$.
M. S. Knebelman (Pullman, Wash.).

Bompiani, E. Una questione sui doppi sistemi coniugati. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 128–132 (1941). [MF 4797]

Let $x_i = x_i(u, v)$ $(i = 0, 1, \cdots, n)$ be projective homogeneous coordinates and suppose that for all i

$$\frac{\partial^2 x_i}{\partial u \partial v} + a\frac{\partial x_i}{\partial u} + b\frac{\partial x_i}{\partial v} + cx_i = 0, \quad a,\ b,\ c \text{ functions of } u,\ v.$$

Then the points x determine a net R. The Laplace transforms $Tx = \partial x/\partial u + bx$, $T^{-1}x = \partial x/\partial v + ax$ determine two new nets TR, $T^{-1}R$, the Laplace transforms of the net R. In a similar way the nets $T^n R$, $T^{-n}R$ $(n > 1)$ are defined. The author proves: If the lines joining the points x, $T^2x = T(Tx)$ determine a W congruence (in other words, if the homogeneous coordinates of these lines satisfy a linear equation of second order), the points x and the net R belong to a space of $n = 3$ dimensions. *G. Fubini* (Princeton, N. J.).

Ermolaev, L. Une classification des correspondances ponctuelles biunivoques entre les surfaces analytiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 425–427 (1941). [MF 4837]

Classifications of one-to-one point correspondences between two surfaces S and Σ are given in terms of involutions and other projectivities in the pencils of tangent lines to S and Σ at corresponding points. The notation and methods are those used by the author previously [the same C. R. **26**, 735–737 (1940); these Rev. **2**, 162]. *V. G. Grove.*

Levy, Harry. Conformal invariants in two dimensions II. Časopis Pěst. Mat. Fys. **69**, 118–127 (1940). (English. Czech and German summaries) [MF 5204]

[For part I, see the same journal **69**, 50–56 (1940); these Rev. **1**, 273.] Der Verfasser verallgemeinert die Resultate seiner vorigen gleich benannten Arbeit auf die Konforminvarianten der Kongruenzkurven auf einer Fläche. Er findet ein System von Dichten auf, welche absolut Konforminvarianten sind, untersucht ihre gegenseitige Beziehungen und benützt die erhaltenen Resultate, um das Problem der Equivalenz von zwei Kongruenzen zu lösen. *Author's summary.*

Cotton, Émile. Sur le calcul des invariants différentiels euclidiens d'une surface. J. Math. Pures Appl. (9) **19**, 211–220 (1940). [MF 4621]

Applications of Cartan's "méthode du repère mobile" in differential geometry for problems as (1) given a surface in parametrical representation, to determine its equation $z = f(x, y) = \sum f_\alpha(x, y)$ ($f_\alpha(x, y)$ a homogeneous form of degree α), taking as x-y-plane the tangent plane of the surface in a given point; (2) to derive the Ricci conditions for a Riemannian manifold V_n which can be imbedded in an Euclidean space E_{n+1}. *H. Samelson* (Princeton, N. J.).

Wong, Yung-Chow. On the generalized helices of Hayden and Sypták in an N-space. Proc. Cambridge Philos. Soc. **37**, 229–243 (1941). [MF 4957]

This paper concerns itself with curves C in a Riemannian space V_n of the following types:

$$\begin{array}{llll} (A)_{2m}: & k_{2s-1}k_{2s}^{-1} = \text{const.}, & s = 1, 2, \cdots, m; & h > 2m; \\ (\bar{A})_{2m}: & k_{2s-2}k_{2s-1}^{-1} = \text{const.}, & s = 1, 2, \cdots, m; & h > 2m-1; \\ (B)_p: & k_{s-2}k_{s-1}^{-1} = \text{const.}, & s = 3, 4, \cdots, p; & h > p-1; \\ (\bar{B})_p: & k_{s-1} = \text{const.}, & s = 2, \cdots, p; & h > p-1, \end{array}$$

wherein $k_0 = 1, k_1, k_2, \cdots, k_{h-1}$ are the curvatures of C and k_h is the first vanishing curvature. Let $\overset{k}{i}_0, \overset{k}{i}_1, \cdots, \overset{k}{i}_{h-1}$ be the components of the tangent vector and of the principal normal vectors. A curve C is called a helix of order $2m+1$ if it admits an auto-parallel vector v^k along it lying in the osculating space $\overset{k}{i}{}_0^{|1}, \overset{k}{i}{}_1^{|2}, \cdots, \overset{k}{i}{}_{h-1}^{|h|}$ and making a constant angle with $\overset{k}{i}_0, \overset{k}{i}_2, \cdots, \overset{k}{i}_{2m}$. Typical theorems are: A helix of order $2m+1$ with $h = 2m+1$ is an $(A)_{2m}$; and conversely an $(A)_{2m}$ with $h = 2m+1$ is a helix of order $2m+1$. There are no helices of order $2m+1$ with $h = 2m+2$. A helix of order $2m+1$ with $h = 2m+3$ is a helix of order $2m+3$. If a helix of order $2m+1$ is an $(A)_{2(m+p)}$, then it is a helix of order $2(m+p)+1$. The latter part of the paper concerns itself with helical curves in an Euclidean space R_n, the projection of such curves on the space R_{n-1} perpendicular to v^k, and finally the spherical image of a curve of type $(A)_{2m}$. [See H. A. Hayden, Proc. London Math. Soc. (2) **32**, 337–345 (1931); M. Sypták, C. R. Acad. Sci. Paris **198**, 1665–1667 (1934); these Rev. **2**, 302.] *V. G. Grove.*

Myers, S. B. Riemannian manifolds with positive mean curvature. Duke Math. J. **8**, 401–404 (1941).

The mean curvature of a Riemannian manifold at a point P for the direction (or vector) v is defined as the sum of the $n-1$ 2-dimensional curvatures obtained by pairing v with each of a set of $n-1$ mutually orthogonal vectors all orthogonal to v (the sum is independent of the choice of such a set). It is shown that a complete Riemannian manifold, which has mean curvature not less than $e^2 > 0$, is closed and has diameter not greater than $e^{-1}\pi(n-1)^{\frac{1}{2}}$. This theorem is a generalization of the corresponding theorem, when mean curvature is replaced by (2-dimensional) curvature [cf. S. B. Myers, Duke Math. J. **1**, 42–43 (1935)], the proof again being based on the consideration of the conjugated points on a geodesic. An example shows the existence of manifolds which have the mean curvature but not curvature bounded away from 0. *H. Samelson* (Princeton, N. J.).

Cartan, Elie. Sur des familles d'hypersurfaces isoparamétriques des espaces sphériques à 5 et à 9 dimensions. Univ. Nac. Tucumán. Revista A. **1**, 5–22 (1940). [MF 4052]

A family of hypersurfaces $V(x^1, \cdots, x^n) = \text{const.}$ of a Riemann space is said to be isoparametric if the first two differential parameters $\Delta_1 V$ and $\Delta_2 V$ of the function V are functions of V. Levi-Civita determined these families of hypersurfaces in the Euclidean space of three dimensions; this determination was extended by Segre to the Euclidean space of any number of dimensions and later by Cartan to non-Euclidean spaces of constant negative curvature. The present paper is based on a method previously given by Cartan for finding in an elliptic or spherical space of $n = p\nu + 1$ dimensions the families of isoparametric hypersurfaces having p distinct principal curvatures of the same degree of multiplicity ν. References are given to the above and certain other of the earlier papers on this subject. In the present paper he deals with the case $p = 4$. It is stated that this case is possible only if $\nu = 1$ or 2, corresponding to $n = 5$ or 9. The geometrical side of the investigation is emphasized. *T. Y. Thomas* (Los Angeles, Calif.).

Yano, Kentaro et Mutô, Yosio. Sur la théorie des espaces à connexion conforme normale et la géométrie conforme des espaces de Riemann. Proc. Imp. Acad. Tokyo **17**, 87–94 (1941). [MF 4855]

This paper is a brief resumé of conformal curve and subspace theory as developed by the authors and others on the basis of Veblen's formalism for conformal geometry. The

Frenet equations, the connection induced on a subspace, the Gauss-Codazzi equations and the equations of generalized circles and auto-concurrent curves are exhibited. Seven new theorems are stated with proofs to be published elsewhere. *J. L. Vanderslice* (Bethlehem, Pa.).

Coburn, N. Conformal unitary spaces. Trans. Amer. Math. Soc. **50**, 26–39 (1941). [MF 4867]

A study is made of the relations between unitary spaces of n dimensions whose fundamental tensors are related by conformal transformations. It is shown that two unitary spaces both without torsion (symmetric connection) cannot be conformal and that if two unitary spaces are conformal then their paths correspond. A theorem for two spaces to be conformal is stated which involves among other conditions the vanishing of a certain curvature tensor. The paper concludes with a study of some particular conformal unitary spaces. *T. Y. Thomas* (Los Angeles, Calif.).

de Mira Fernandes, A. Un vettore ausiliare in analisi tensoriale. Portugaliae Math. **2**, 139–144 (1941). [MF 4823]

A vector field u_i, $i=1, 2, \cdots, n$, is assumed in an n-dimensional manifold with coordinates x^i, and tensors are differentiated with respect to this vector field. The author writes $\partial T_{ii}/\partial u_l = T_{ii}^{|l}$. The result has again tensor character. Vector u_i is called a vector of characteristic n. This principle is applied to a number of special cases in affine and metrical manifolds. *D. J. Struik* (Cambridge, Mass.).

Michihiro, Satoshi. A remark to a covariant differentiation process. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 189–192 (1940). [MF 4144]

A process for the covariant differentiation of tensors along curves is given which contains as a special case the formulas of Craig [Bull. Amer. Math. Soc. **37**, 731–734 (1931)] and M. M. Johnson [ibid. **46**, 269–271 (1940); these Rev. **1**, 273]. *T. Y. Thomas* (Los Angeles, Calif.).

Doyle, T. C. Tensor decomposition with applications to the contact and complex groups. Ann. of Math. (2) **42**, 698–722 (1941). [MF 4969]

In the first part of this paper the author shows how an arbitrary invariant (tensor) in the phase space of N dimensions may be completely decomposed into factors in complementary subspaces of q and $N-q$ dimensions. This is accomplished by means of a factor frame, which is a nonsingular matrix, in a manner somewhat more general than that used by L. P. Eisenhart and the reviewer in the study of contact transformations. This factoring, however, is still open to the objection that the invariant theory is valid only for a subgroup (Σ) of the full group of analytic transformations in the phase space and this subgroup is determined by the factor frame. With various specializations the paper gives the invariant theory under the contact group, the complex group and the geometric setting of Schouten's doubly homogeneous contact transformation. A metric is introduced [§ 14] in the most natural manner and it is then shown that for this metric the paths and geodesics coincide. *M. S. Knebelman* (Pullman, Wash.).

Wade, T. L. Tensor algebra and Young's symmetry operators. Amer. J. Math. **63**, 645–657 (1941). [MF 4690]

After a discussion of Young's symmetry operators so-called immanant tensors are introduced, which are certain numerical tensors associated with each partition of a natural number p. These immanant tensors are used for the decomposition of a tensor into its irreducible components. Among the many references, we miss Chapter VII of Schouten's "Ricci-Kalkül." *D. J. Struik* (Cambridge, Mass.).

Wade, T. L. A note on subgeometries of projective geometry as the theories of tensors. Bull. Amer. Math. Soc. **47**, 475–478 (1941). [MF 4540]

That the group of a subgeometry of projective geometry leaves a certain form invariant means, in tensor language, that a certain numerical tensor is covariant. It is pointed out that the latter interpretation and the use of tensor algebra frequently furnish the algebraic invariants more quickly than the usual theory. As examples the author derives an algebraically complete system of Euclidean invariants for the ternary cubic curve, and gives a simple proof for a result of Weitzenböck on the invariants in Galilei-Newton geometry. *H. Busemann* (Chicago, Ill.).

Chern, Shiing-shen. Sur les invariants intégraux en géométrie. Sci. Rep. Nat. Tsing Hua Univ. (A) **4**, 85–95 (1940). [MF 4529]

The author investigates invariant measures of sets of geometric elements in a Klein space. He uses the moving reference frame method of Cartan and exhibits the possible measures as invariant integrals with integrands expressed in terms of the relative components of the moving frames. Two examples are worked out: the parabolas of the unimodular affine plane and the oriented circles of the Moebius plane. *J. L. Vanderslice* (Bethlehem, Pa.).

Suguri, Tuneo. The geometry of K-spreads of higher order. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **1**, 143–166 (1941). [MF 3978]

Douglas has shown how the general geometry of paths in an n-dimensional coordinate space S_n may be developed by a study of the various groups of coordinate and parameter transformations which are allowable in the geometry. He also showed that these ideas may be applied to develop a geometry of systems of K dimensional manifolds S_K in S_n [Ann. of Math. (2) **29**, 143–168 (1928) and Math. Ann. **105**, 707–733 (1931)]. In the present paper an analogous investigation is made under the following hypotheses: (1) The allowable transformations are the so-called rheonomic transformations. If the equations of the S_K in S_n are $x^i=x^i(u^a)$, $i, j=1, 2, \cdots, n$; $a=1, 2, \cdots, K$, these transformations take the form $\bar{x}^i=\bar{x}^i(x^j, u^a)$, $\bar{u}^a=u^a$. (2) The system of S_K's is defined by differential equations of the mth order with $m\geqq 2$. A detailed geometry is developed which includes definitions of the coefficients of connection, covariant differentiation, curvatures and torsions for the S_K. If $K=1$, the results for the rheonomic geometry of curves are obtained [cf. Hombu's paper reviewed on p. 20]. *A. Fialkow.*

Seetharaman, V. On the existence of a metric for path-spaces of order two. Proc. Indian Acad. Sci., Sect. A. **12**, 399–406 (1940). [MF 3763]

Let $H_i=0$ represent the differential equations of the extremals of $\delta\int f dt=0$. Then the equations of variation of H_i are necessarily self-adjoint. If the order of H_i is two this condition is sufficient for the existence of f [D. R. Davis, Bull. Amer. Math. Soc. **35**, 371–380 (1929)]. Following a path broken by D. D. Kosambi [Quart. J. Math., Oxford Ser. **6**, 1–12 (1935)], the author in this paper extends Davis' result to equations of order three. He finds that self-adjointness implies $f=\varphi_i\ddot{x}^i+\psi$, with ϕ_i known. The main problem is to show that integrability conditions for the

existence of ψ are also consequences of this condition. Its solution requires full use of tensor methods.

J. L. Vanderslice (Bethlehem, Pa.).

Shabbar, Mohammad. On the existence of a metric for path-spaces admitting the Lorentz group. Proc. Indian Acad. Sci., Sect. A. **13**, 203–210 (1941). [MF 4765]

This paper is an outgrowth of investigations of D. D. Kosambi. The author takes Kosambi's form of path equations admitting the Lorentz group and applies to them the general condition that a set of paths be the extremals of a variation problem, namely, that their equations of variation be self-adjoint. A study of the resulting differential equations gives rise to a set of necessary and sufficient conditions for the existence of a metric and a method of solution. The same method of attack is then applied to more specialized paths and metric, leading to simpler conditions and verifying an unpublished result of Kosambi.

J. L. Vanderslice (Bethlehem, Pa.).

Chern, Shiing-shen. The geometry of higher path-spaces. J. Chinese Math. Soc. **2**, 247–276 (1940). [MF 4212]

The subject is the geometry of the system of equations

$$\frac{d^r x^i}{dt^r} + F^i\left(\frac{d^{r-1}x}{dt^{r-1}}, \cdots, \frac{dx}{dt}, x, t\right) = 0$$

under the group $\bar{x}^i = \bar{x}^i(x, t)$, $\bar{t} = t$. Diverging from the methods employed on this and similar problems by Kawaguchi, Kosambi and others, the author treats it as a problem in equivalence by a very lucid application of the methods of E. Cartan. The given equations are regarded as a Pfaffian system of rn equations in $rn+1$ variables and then replaced by rn linear combinations with auxiliary variables as coefficients, in terms of which the equivalence problem is stated. Invariant differential conditions placed successively on the auxiliary variables eliminate all but n^2 of them and "equations of structure" arise expressing the differentials of the Pfaffians as linear combinations of the Pfaffians themselves. This exhibits the theory as a generalized geometry in the sense of Cartan in $rn+1$ dimensions. The n^2 free auxiliary variables correspond to freedom of choice in the local reference frames and can be chosen to reduce the notation to tensor form. A complete system of invariants arises from the coefficients (curvature and torsion quantities) of the equations of structure and their covariant derivatives, of which there are $r+1$ distinct types. For the cases $r=2, 3$, explicit formulas are given for curvature, torsion and the several processes of covariant differentiation. A special section is devoted to the case in which the transformations of x are independent of t. Finally consequences of the vanishing of curvature and torsion are discussed.

J. L. Vanderslice (Bethlehem, Pa.).

Hombu, Hitoshi and Okada, Kazuo. On the projective theory of asymmetric connections. Proc. Phys.-Math. Soc. Japan (3) **23**, 357–362 (1941). [MF 4771]

Let Γ^i_{jk} be an arbitrary asymmetric affine connection defined on an n-dimensional space. For reasons which are explained in the paper, the authors define a projective change of the affine connection as a transformation of the form

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \phi_j\delta^i_k + \psi_k\delta^i_j,$$

where ϕ_j and ψ_k are arbitrary covariant vectors and the δ^i_j are the Kronecker deltas. [It should be noted that these transformations preserve paths but are not the most general changes of an asymmetric connection with this property.] It is shown that a formal theory based on these changes of connection can be developed in a manner entirely analogous to the projective theory of T. Y. Thomas [Differential Invariants of Generalized Spaces, Cambridge, England, 1934, chap. 3]. In particular, coefficients of a projective connection and the equi-projective curvature tensor are defined. As in the work of Thomas, it is possible to reduce the problem of equivalence of the projective connections of this paper to a restricted problem for affine connections in an associated manifold of $(n+1)$ dimensions.

A. Fialkow (Brooklyn, N. Y.).

Hombu, Hitoshi. On the geometry of paths of higher order. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **1**, 129–142 (1941). [MF 3977]

A transformation of a parameter t and of the n coordinates x^i of a space is called rheonomic if it is of the form $\bar{x}^i = \bar{x}^i(x^j, t)$, $\bar{t} = t$; $i, j = 1, 2, \cdots, n$. The equivalence problem of general paths of the second order under rheonomic transformations was discussed by E. Cartan [Math. Z. **37**, 619–622 (1933)] by means of an affine connection. In Part I of the present paper the author develops the fundamentals of the rheonomic geometry of paths of the third order by means of an intrinsic affine connection of the extended tangent affine space. In Part II the method of a previous paper by A. Kawaguchi and the author [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **6**, 21–62 (1937)] is employed to obtain still another treatment of the rheonomic geometry of paths of order $m \geqq 3$. *A. Fialkow* (Brooklyn, N. Y.).

Pinney, Edmund. General geodesic coordinates in a general differential geometry. Tôhoku Math. J. **47**, 111–120 (1940). [MF 4012]

Michal and Hyers [Ann. Scuola Norm. Super. Pisa **7**, 157–176 (1938); Math. Ann. **116**, 310–333 (1939)] studied a type of geodesic coordinates, called abstract normal coordinates, in a general differential geometry with abstract normed linear coordinates [see Michal, Bull. Amer. Math. Soc. **45**, 529–563 (1939); cf. these Rev. **1**, 29; references are given there to the earlier work of Michal]. Under the assumption of the existence of the $(r-1)$st successive Fréchet differential of the components of the abstract linear connection, the author proves the existence of general geodesic coordinates of order r. The author's assumptions on the abstract linear connection and on the transformations of abstract coordinates are essentially those found in the earlier work of Michal, and hence are less restrictive than those of Michal-Hyers for abstract normal coordinates. In the development of his theory, the author makes an important application of some of R. S. Martin's results [California Institute of Technology thesis, 1932, much of which is still unpublished] on abstract polynomials and power series expansions. The whole paper is essentially a generalization of a paper by Michal [Bull. Amer. Math. Soc. **36**, 541–546 (1930)] on finite dimensional differential geometry.

A. D. Michal (Pasadena, Calif.).

Kawaguchi, Akitsugu. Die Differentialgeometrie höherer Ordnung II. Über die n-dimensionalen metrischen Räume mit vom m-dimensionalen Flächenelement abhängigem Zusammenhang. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 153–188 (1940). [MF 4143]

[The first part appeared in the same J. **9**, 1–152 (1940); cf. these Rev. **2**, 22.] In the metric space X_n the metric tensor is made to depend on r contravariant vector fields.

Conditions are imposed so that it is dependent on the linear vector space V^r. [Some of these conditions are redundant.] Covariant differentiation of an invariant is defined and, since differentiation with respect to any vector leads to an invariant, there are numerous sets of relations between these derivatives. The author gives 117 sets of them.

M. S. Knebelman (Pullman, Wash.).

Devisme, Jacques. Sur un espace dont l'élément linéaire est défini par $ds^3=dx^3+dy^3+dz^3-3dxdydz$. J. Math. Pures Appl. (9) 19, 359–393 (1940). [MF 4629]

In his 1933 Paris thesis [Ann. Fac. Sci. Univ. Toulouse 25, 143–238 (1933)], the author studied the differential equation $U_{xxx}+U_{yyy}+U_{zzz}-3U_{xyz}=0$. Associated with this equation is a three-dimensional space S with element of arc length $ds^3=dx^3+dy^3+dz^3-3dxdydz$. In the present paper the author continues his study of the elementary geometry of the space S. Considerable attention is also given to the theory of the Appell moving trihedral, an analogue of the well-known moving trihedral of Euclidean geometry. Extensive use is made of Appell's three special functions $P(\phi, \theta)$, $Q(\phi, \theta)$, $R(\phi, \theta)$. The rôle of these functions in the theory is analogous to that played by the direction cosines in 3-dimensional Euclidean geometry. [See P. Appell, C. R. Acad. Sci. Paris 84, 540 and 1378 (1877).] The paper is formal and no attempt is made to give the foundations of the geometry.

A. D. Michal (Pasadena, Calif.).

Cartan, E. The geometry of differential equations of third order. Revista Mat. Hisp.-Amer. (4) 1, 3–33 (1941). (Spanish) [MF 4723]

Let x, y vary in a plane P, and X, Y, Z in a space E. In general, an equation (1) $f(x, y, X, Y, Z)=0$ determines ∞^2 surfaces S_{xy} depending on two parameters x, y, and ∞^3 curves C_{XYZ} depending on three parameters X, Y, Z. The curves C are the integrals of a differential equation Γ of third order; the surfaces S are determined by another system Σ of differential equations. An equation is said to be equivalent to (1) if it can be deduced from (1) by means of a point transformation in the plane P and another point transformation in E. The author finds some relations between the equations (1), Γ, Σ; his main problem is to determine when two given equations (1) are equivalent. He considers the corresponding equations Γ of third order and shows that, given an arbitrary equation Γ of third order, a corresponding space of linear isotropic elements can be found such that the corresponding spaces are isomorphic if and only if two equations Γ correspond to equivalent equations (1). In studying this problem Cartan makes use of his method of Pfaffians, which is important for many modern geometric theories. He studies, first of all, the case when the equation Γ is $y'''=0$, so that (1) becomes $y=\frac{1}{2}x^2X+xY+Z$; in this case the surfaces S_{xy} are the isotropic planes of a pseudo-Euclidean space E, for which $Y^2-2XZ=0$ is the equation of the isotropic cone, with the vertex at the origin. These planes are a complete integral of the equation $P+\frac{1}{2}Q^2=0$ (where $P=\partial Z/\partial X$, $Q=\partial Z/\partial Y$). The system Σ is the system $2dZ+Q^2dX-2QdY=0$; $dQ=0$. To every linear element of second order (x, y, y', y'') of P corresponds an isotropic linear element of E. The author develops the geometry of the space of these elements by means of his method of Pfaffians, and, by means of the same methods, generalizes the results which led him to the geometries of connection. Many interesting particular cases are studied.

G. Fubini (Princeton, N. J.).

Chern, Shiing-shen. The geometry of the differential equation $y'''=F(x, y, y', y'')$. Sci. Rep. Nat. Tsing Hua Univ. (A) 4, 97–111 (1940). [MF 4528]

The geometry (invariant theory) of the equation $y'''=F(x, y, y', y'')$ under the group of contact transformations is formulated as a problem in the equivalence of Pfaffian systems (the method of Cartan). A certain differential invariant I of F is fundamental, the invariant theory being quite distinct in the two cases $I=0$, $I\neq 0$. The following geometric interpretations arise: when $I=0$ the space of the integral curves is conformally connected; when $I\neq 0$ it is a generalized space with a certain five parameter fundamental group.

J. L. Vanderslice (Bethlehem, Pa.).

MECHANICS

Dynamics, Celestial Mechanics

Wintner, Aurel. On the problem of analyticity in dynamics. Proc. Nat. Acad. Sci. U. S. A. 27, 311–314 (1941). [MF 4580]

The author constructs an interesting example of a function $F(x, y)$ which is continuous for all (x, y), which is periodic with period 2π in each of the variables x and y, such that neither $F(x, a)$ nor $F(a, y)$, a fixed, is absolutely continuous, such that $F(x, y)$ is not absolutely continuous, and yet such that $f(t)=F(t-t_0, \lambda t)$, where t_0 is arbitrary and λ is a fixed irrational number (upon which the definition of F depends), is analytic on the whole t-axis. The function $F(x, y)$ is defined by the series

$$\sum_{k=1}^{\infty} n_k^{-1} \cos (m_kx-n_ky),$$

where m_k and n_k are integers such that $n_k>k^2$ and $|m_k-n_k\lambda|<1/n_k$, $k=1, 2, \cdots$. Due to the irrationality of λ, the set of points $(t-t_0, \lambda t)$ corresponding to the t-axis is everywhere dense in the (x, y) plane reduced mod $(2\pi, 2\pi)$. The author has constructed this simple example to show that the limiting process leading from the general solution to an integral in the large in dynamics need not preserve the analytic character given with the flow.

G. A. Hedlund.

de Mira Fernandes, A. Sistema derivato di un sistema dinamico. Portugaliae Math. 2, 162–165 (1941). [MF 5043]

Given a system S of moving mass points $P_1(m_1), \cdots, P_N(m_N)$; let $Q_i=P_i+P_i'$, where P_i' denotes the velocity of P_i, and associate the mass m_i with the point Q_i. The "derived system" S' is by definition the system of moving points $Q_i(m_i)$, $i=1, \cdots, N$. Several theorems are proved about the relationship between S and S'. For example, the center of gravity of S' moves as if it were a free point of mass $\sum_{i=1}^{n} m_i$ under the action of the vectorial sum of all the external forces acting on S and of their time derivatives. The case when S is a solid is also considered.

D. C. Lewis (Durham, N. H.).

Agostinelli, Cataldo. Moto di due corpi rigidi collegati in un punto e di cui uno ha un punto fisso. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 75, 249–263 (1940). [MF 4777]

Let S_1 and S_2 be two rigid bodies, O a fixed point of S_1,

and S_2 rigidly connected to S_1 at a point Q. Assuming no friction, the equations of motion are obtained and discussed in two cases, that in which no forces act on the system $S=S_1+S_2$, and that in which Q is the center of gravity of S_2 while the system S is subject only to the action of gravity. In the latter case it is shown that S_1 rotates about O like a rigid body consisting of S_1 plus the mass of S_2 concentrated at the point Q, and the permanent rotations of S_1 about O are discussed. *M. C. Gray* (New York, N. Y.).

Chazy, Jean. Oscillations isochrones dans un mouvement où la force dépend seulement de la position. C. R. Acad. Sci. Paris **211**, 621–624 (1940). [MF 5366]

The equation of motion of a material point on a straight line when the force depends on the position only is $mx''=X(x)$. In the case of oscillatory motion between the limits a and b, choose the origin of x in a point of equilibrium on the segment ab, which will always exist if $X(x)$ is continuous for $a<x<b$. The author defines u, α and $f(u)$ by

$$(2/m)\int_0^x X(x)dx=-u^2,\quad 2h/m=\alpha^2,\quad dx/du=f(u),$$

whence the period of oscillation is

$$T=2\int_{-\alpha}^{+\alpha} f(u)du/(\alpha^2-u^2)^{\frac{1}{2}}.$$

The elementary case of tautochronous motion, in which the attractive force is directly proportional to the distance x, is $f(u)=$constant. The motion is still tautochronous if $f(u)$ is the sum of a constant and any arbitrary odd function of u. Two examples are given. *D. Brouwer.*

Chazy, Jean. Sur une loi corrective de la loi de Newton. J. Math. Pures Appl. (9) **19**, 261–280 (1940). [MF 4624]

A more detailed presentation of the contents of two earlier notes by the same author [C. R. Acad. Sci. Paris **209**, 133–136 (1939); **210**, 713–716 (1940); these Rev. **1**, 20; **2**, 264]. *D. Brouwer* (New Haven, Conn.).

Neronoff, N. Sur une extension de la loi de l'attraction de Newton. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 35–43 (1940). (Russian. French summary) [MF 4747]

The author is concerned with the determination of possible trajectories of a planet (considered as a point of mass m_1) about the sun (considered as a sphere of mass m_2, and whose density depends only on the radius of the sun) under the law of attraction characterized by potential of the form

$$U=m_1m_2\sum_{i=1}^{\infty} b_i/\rho^i,$$

where the b_i are constants and ρ is the distance. The problem is shown to be equivalent to obtaining the integral of the equation of motion of a system with one degree of freedom,

$$\ddot{x}+x=a_2x^2+a_3x^3+\cdots,$$

in the vicinity of the position of stable equilibrium ($x=0$), under the boundary conditions $x(0)=c$, $\dot{x}(0)=0$. *I. S. Sokolnikoff* (Madison, Wis.).

Goldsbrough, G. R. The theory of the divisions in Saturn's rings. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 183–216 (1941). [MF 4876]

The problem of the stability of Saturn's rings and the cause of the divisions in them has been of interest to mathematical astronomers since the phenomena were first observed. The present paper is an examination of the problem by the methods of periodic orbits. Only the disturbances created in a ring of the infinitesimal bodies by a moon of Saturn is discussed. The satellite is supposed to be of mass m', which is small compared to that of Saturn (M) and rotates in a circle about Saturn. The author supposes that in a circle of radius a about Saturn there are P infinitesimal particles which are in equilibrium at the vertices of a regular p-gon, when $m'=0$. The differential equations of variation of these particles for an arbitrary small displacement and for $m'\neq 0$ are set up and integrated in power series in m'. The arbitrary constants of the solution are adjusted so that the series are periodic. They are convergent if m' is sufficiently small for all values of t. The author finds from the above discussion that the Satellite Mimas is the only one which causes the divisions in the rings.

It would be of interest if the author could have discussed the upper limit of the size of m' that would insure convergence. This limit might be so much smaller than Mimas that his theory would not apply. One must note that several other approximations are made. Wherever $M+m$ or $M+m'$ occurs each is replaced by M. Equations (52) and (54) are solved by approximations. Again it should be observed that, while periodic orbits for the particles are constructed, there may be no initial conditions in the actual problem which will be the same as the initial conditions in the theory. The author notices this weakness but his explanation does not carry conviction. *H. E. Buchanan* (New Orleans, La.).

Järnefelt, G. Zur relativistischen Perihelbewegung der Planetenbahnen. Ann. Acad. Sci. Fennicae (A) **53**, no. 2, 32 pp. (1940). [MF 5249]

Hydrodynamics, Aerodynamics, Acoustics

***Theodorsen, Theodore. Impulse and momentum in an infinite fluid.** Theodore von Kármán Anniversary Volume, pp. 49–58. California Institute of Technology, Pasadena, Calif., 1941.

The author studies impulse and momentum relations in an infinite region of fluid when a body is moving in it. The momentum of fluid enclosed between an inner boundary and an outer boundary can be calculated as the impulse exerted on the liquid at the origin minus the impulse exerted by the outer boundary. If the surface of the moving body is taken as the inner boundary, the impulse exerted on this boundary is usually called the "Kelvin impulse." By choosing the outer boundary so that its dimension in the direction of momentum is infinitely large compared with the dimensions in other directions, the impulse exerted on the outer boundary is zero. Therefore, it can be said that all the momentum in this particular direction is contained in this outer boundary. Any other shape of the outer boundary gives a finite impulsive pressure and therefore only part of the momentum is contained in this outer boundary. *H. S. Tsien* (Pasadena, Calif.).

Preston, J. H. Asymptotic solution of Southwell and Squire; modification to Oseen's equations. Philos. Mag. (7) **31**, 413–424 (1941). [MF 4700]

Southwell and Squire [Philos. Trans. Roy. Soc. London.

Ser. A 232, 27–64 (1933)] have proposed a modification to Oseen's approximation to the equations of motion of a viscous incompressible fluid. Instead of linearizing the equations by replacing u, v when these occur explicitly by their values at infinity, that is, $U, 0$, they replace them by u', v', the velocities of inviscid flow past the cylinder under consideration. They then replace the rectangular coordinates x, y by the orthogonal coordinates α, β, the velocity potential and stream function for the irrotational flow. In the present paper an asymptotic solution of the resulting equations is obtained for large Reynolds numbers by a method already applied to a similar problem by Piercy and Winny [Philos. Mag. (7) 16, 390–408 (1933)]. The solution is used to calculate the skin friction for elliptical cylinders. The results, including the separation point, are all in agreement with other approximate boundary-layer theories, and their qualitative agreement with experiment confirms Southwell and Squire's belief that their modification of Oseen's equations should give an indication of the changes of flow pattern with Reynolds number, differing but little from Oseen's approximation at low Reynolds numbers. *W. R. Sears.*

Broïkos, Ath. Sur le mouvement discontinu d'un fluide limité par un paroi fixe et une ligne libre. Pont. Acad. Sci. Comment. 3, 627–657 (1939). [MF 4105]

General solution of the problem of plane irrotational flow of an incompressible fluid limited on one side by a solid wall μ and on the other by a line of discontinuity λ separating the flowing fluid mass from one at rest; the flow is obstructed by a solid cylindrical obstacle of closed cross-section. The assumptions and the methods of solution are those adopted in various investigations of somewhat less general scope by Villat, Levi-Civita and Cisotti. The case that the surface μ is a plane is considered in greater detail.
P. Nemenyi (Fort Collins, Colo.).

Flügge-Lotz, I. und Ginzel, I. Die ebene Strömung um ein geknicktes Profil mit Spalt. Ing.-Arch. 11, 268–292 (1940). [MF 4219]

The authors determine the function which maps a rectangle into the exterior of two straight intervals, the directions of which form a given angle β. As is well known, the approximate determination of a two-dimensional potential flow in the case where the cross-section consists of two disconnected profiles can be reduced to the above mathematical problem. Using elliptic functions the authors obtain the mapping function in a closed form, a result previously obtained by Garrick [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 542 (1935)] and Bonder [Conformal mapping of the exterior of two arbitrary circular arcs into an annulus, Warsaw, 1935]. By combining this mapping with other simple mappings solutions are obtained in the case of a wing and rudder of finite thickness. Several examples are given; the distribution of the pressure and the influence of the rudder are discussed. In the introduction the authors give a short historical sketch which, in the opinion of the reviewer, requires corrections. They do not mention results of certain other writers closely connected with the problem considered. For example, the above cited and various other papers by Bonder, the papers of Villat, etc., are not indicated. Bonder [Rep. Polytech. Soc. Warsaw 4 (1925)] gave the solution for a flow in the exterior of two cylinders in the form of an infinite series which, as the reviewer has indicated [Bergmann, Z. Angew. Math. Mech. 9, 244 (1929)], can be written in a closed form in terms of elliptic functions.
S. Bergman (Providence, R. I.).

Ferrari, Carlo. Il problema dell'elica con vento laterale. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 75, 338–368 (1940). [MF 4781]

The author shows how one can find the axial and normal increments of velocity produced by an airscrew, whose axis forms any angle α with the direction of the asymptotic incident flow, and so by Pistolesi's method of mean increments the aerodynamic characteristics of the airscrew itself. The method used is that of the potential of acceleration in the form indicated by von Kármán and Burgers. The results now obtained agree with those found previously for the case of small inclination α, while for any angle α the corrective terms can be found. In an appendix some of the expressions used are found by contour integration. *H. Bateman.*

Ferrari, Carlo. Sulla determinazione del proietto di minima resistenza d'onda. I. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 74, 675–693 (1939). [MF 4774]

The author's aim is to find the form of the solid of revolution of least wave resistance for a given ratio of maximum breadth to length. The motion is supposed to be supersonic translatory motion in the direction of the axis and the solid is treated as so thin that linear equations can be used as in the work of von Kármán. Use is made of Hankel functions of order one. The formula for the drag coefficient is quite analogous to that found by von Kármán. The minimum problem leads to an integral equation which is replaced by the simpler equation

$$\int_{-1}^{1} \frac{\gamma(\alpha_2)}{\alpha_2-\alpha_1} d\alpha_2 = \lambda_1 + \lambda_2 F_2(\alpha_1),$$

where

$$F_2(\alpha_1) = 1 - 2(\alpha_1 - a)/[(\alpha_1-a)^2 - c^2r^2]^{\frac{1}{2}}.$$

The equation is solved by assuming that

$$\gamma = A_0 \cot\theta - \tfrac{1}{2}A_1 \operatorname{cosec}\theta \cos 2\theta + \sum_{n=2}^{\infty} A_n \sin n\theta,$$

where $\alpha_2 = -\cos\theta$. *H. Bateman* (Pasadena, Calif.).

Ferrari, Carlo. Sulla determinazione del proietto di minima resistenza d'onda. II. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 75, 61–96 (1939). [MF 4775]

The force exerted on the solid is found from the loss of kinetic energy, which is equal to the pressure produced by the shock wave generated by the shot. The author's aim is to find the form of wave which makes the force a minimum. Let V_1 be the fluid's velocity immediately in front of the wave and V_1' the velocity which the fluid would acquire if, after the shock, it expanded adiabatically to the former pressure p_1. It is found that

$$1-(V_1'/V_1)^2 = [2/(k-1)] \sin^2\alpha[e^{\Delta s/kC_v}-1],$$

where k is the adiabatic index, α is Mach's angle for velocity V_1, C_v is the specific heat for constant volume and Δs is the difference of entropy between the states V_1', p_1 and V_1, p_1.

There is also an equation

$$3\Delta s/C_v = 2k(k^2-1)\operatorname{cosec}^3 2\alpha \tan^3\beta(1+3c\tan\alpha\tan\beta+\cdots),$$

where $c = k\operatorname{cosec}^2 2\alpha + \cot^2 2\alpha$ and β is the change in direction of velocity in crossing the wave. The quantity Δs is now eliminated and an expression for the force F obtained in the form

$$F = (\pi/3)(k+1)\sec^3\alpha \operatorname{cosec} 2\alpha\, p_1 V_1^2 \times \int_0^\infty \tan^3\beta[1+3c\tan\alpha\tan\beta+\cdots]yy'dx.$$

By a simplifying hypothesis this is reduced to

$$F=A\int_0^\infty yy'[(y'^2/(1+y'^2))-\sin^2\alpha]^2dx.$$

In the treatment of the variation problem use is made of ideas derived by analogy from those in the corresponding problem in plane motion. Use is also made of Tricomi's results. A method of integration step by step is finally used to find the field of flow and the form of the body when the form of the wave has been found. A numerical example is given. *H. Bateman* (Pasadena, Calif.).

Ferrari, Carlo. Sul problema del proietto di minima resistenza d'onda. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 578–596 (1940). [MF 4789]

The problem treated in note I [cf. the preceding reviews] is treated by another method leading to an integral equation

$$\int_{-1}^{-\sigma} f'(\alpha_2)K(\alpha_1, \alpha_2)d\alpha_2+\lambda_1F_1(\alpha_1)+\lambda_2=0,$$

where $K(\alpha_1, \alpha_2)$ is expressed in terms of elliptic integrals of Legendre's first and third types. The method of solution suggested is one in which the range of integration is divided up into a number of small parts. *H. Bateman.*

Roy, Maurice. Sur la stabilité des ondes de choc orthogonales dans un écoulement par tranches. C. R. Acad. Sci. Paris **212**, 369–371 (1941). [MF 4908]

The author studies the stability of a spherical shock wave in a uniform radial flow of perfect, compressible, nonviscous gas. The disturbance is assumed to be small and only first order terms are retained in the equations. The shock wave is said to be stable if after an initial disturbance it tends to move back to its initial position. It is found that under these assumptions the spherical shock wave is stable if the radial flow is divergent. This result can be applied to the supersonic flow in a nozzle. Then according to the theory, a shock wave perpendicular to the flow direction (straight shock wave) can only appear in the divergent part of the nozzle. *H. S. Tsien* (Pasadena, Calif.).

✱Bollay, William. The theory of flow through centrifugal pumps. Theodore von Kármán Anniversary Volume, pp. 273–284. California Institute of Technology, Pasadena, Calif., 1941.

The problem is to determine the flow around the blades of a centrifugal pump impeller, given the angular velocity of the pump, the amount of rotation in the flow before entering the impeller due to pre-rotation vanes and the total quantity of flow through the impeller. The flow is assumed to be two-dimensional, incompressible, nonviscous, and leaves the trailing edges of blades tangentially. The shape of blades is assumed to be approximately radial. By using a coordinate system fixed to the rotating impeller, each blade is transformed into a circle by conformal representation. The boundary conditions are then (1) zero normal fluid velocity over the boundary of the circle; (2) zero tangential fluid velocity at a point of the circular boundary which represents the trailing edge of the blades. These conditions are satisfied by a distribution of sources and sinks over the boundary of the circle and a vortex at the center of the circle. The final results are expressed in terms of integrals involving the given quantities of the problem. These integrals can be evaluated in terms of elementary functions if the number of blades is equal to 1 or 2, and in terms of elliptical integrals for 4 blades. For larger number of blades, series expansions or other approximate methods have to be used. The same problem is solved by W. Spannhake [see W. Müller, Mathematische Strömungslehre, Springer, Berlin, 1928, p. 219] in terms of infinite series. *H. S. Tsien* (Pasadena, Calif.).

✱Reissner, Hans. On lubrication flow with periodic distribution between prescribed boundaries. Theodore von Kármán Anniversary Volume, pp. 310–316. California Institute of Technology, Pasadena, Calif., 1941.

The unclarified entrance and exit conditions in the known theory of the viscous flow between a wedge shaped Michell pad and the counter surface of a bearing makes it desirable to attack the problem from a different side. The author, therefore, gives a complete theory of the two-dimensional viscous flow produced in a very narrow channel with straight boundary on one side and an arbitrarily but periodically formed boundary for the other side, the flow being produced by the uniform motion of the straight boundary. The inverse problem, that is, determination of the boundary curve pertaining to a prescribed pressure distribution, is also solved. Possible implications of the results for the practical problem of bearings are indicated but it is pointed out that decisive results in this field will have to be given by a coordination of experimental investigation with mathematical theory. *P. Nemenyi* (Fort Collins, Colo.).

Ogibalov, P. M. On the spread of plastico-viscous flow about a rotating cylinder. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, no. 1, 13–30 (1941). (Russian. English summary) [MF 4656]

The author considers the spread of plastico-viscous flow in an unlimited plane about a rotating cylinder with an angular velocity varying according to an arbitrary law. The radius of the domain of flow is found to be a function either of time or of velocity of rotation. This problem is important for the experimental proof of the primary hypothesis of the plastico-viscous flow. *A. Weinstein.*

Surdin, M. Étude du mouvement permanent de rotation de deux sphères rigides dans un liquide visqueux. Portugaliae Math. **2**, 145–152 (1941). [MF 4641]

The author studies the motion of an incompressible viscous fluid around two spheres rotating with a constant angular velocity about the line connecting their centers. The fluid at infinity is at rest. The inertia terms in the dynamical equations are neglected. Furthermore, it is assumed that the velocity component in the direction of axis of rotation is zero and the pressure in the fluid is constant. These assumptions are physically plausible and agree with previous study of the fluid motion around a single rotating sphere. They are justified by the fact that the solution so obtained satisfies all the boundary conditions of the problem. The general solution is carried out with the aid of stream function and bipolar coordinates. Finally the retarding torque on the spheres is calculated as a series. *H. S. Tsien* (Pasadena, Calif.).

Sakadi, Zyuro. On the dispersion of sound wave considering the effects of heat conduction and viscosity. Proc. Phys.-Math. Soc. Japan (3) **23**, 208–213 (1941). [MF 4523]

The fundamental equations for the problem defined by

the title are actually known [cf. appendix G of the reviewer's paper, Phys. Rev. (2) 50, 355–364 (1936)]. The writer includes also the reduction along familiar lines of the expressions for the absorption coefficient and velocity.

D. G. Bourgin (Urbana, Ill.).

✶**Epstein, Paul S. On the absorption of sound waves in suspensions and emulsions.** Theodore von Kármán Anniversary Volume, pp. 162–188. California Institute of Technology, Pasadena, Calif., 1941.

The mathematical problem is that of a periodic acoustic wave incident on a spherical obstacle which is (a) rigid or (b) a viscous fluid or (c) an elastic solid. The surrounding medium is a viscous fluid. The current velocity may be written $v=-\nabla\psi+\nabla\times B$, $\nabla\cdot B=0$, and then the Stokes Navier equations give the first order description of the fluid. Thus

$$(1)\qquad \rho\psi_{tt}=\nabla^2\left(\frac{4}{3}\eta\psi_t+\rho\frac{\partial p}{\partial\rho}\psi\right),\quad \rho B_{tt}=\eta\nabla^2B_t,$$

where η is the coefficient of viscosity and the other notation is standard. Longitudinal waves are given by ψ and transverse waves by B. Suppose $Be^{i\omega t}=A$, $\psi e^{i\omega t}=\Phi$ are independent of t; then ∇^2+k^2 annuls A and Φ for different k values (determined by (1)). It is seen at once from these k values that the transverse waves are much more rapidly damped than the longitudinal waves. This applies both to the medium and to (b). Thus the main energy loss in acoustic reflection is due to partial conversion of the incident longitudinal wave into a rapidly absorbed transverse wave. The mathematical justification follows. Let $\Phi_i=\exp ikr\cos\theta$ be the incident potential along the polar axis of the sphere. The reflected waves are given by Φ_R and $(A_\phi, A_r=0, A_\theta=0)$ and the refracted waves by Φ' and A_ϕ'. On expanding all potentials in series of spherical harmonics and making use of the boundary conditions depending on which of (a), (b), (c) is assumed, a set of recurrence relations between the coefficients in the various series is obtained. Mathematical arguments connected with polynomial approximation to the spherical harmonics and physical arguments involving the size and relative density of the spheres indicate only the first few coefficients in say Φ_R are significant and these are calculated approximately. The dissipation loss averaged over a large sphere is given by $\frac{1}{2}\omega\rho\Re\int(i\bar{f}\partial f/\partial r)dS$. On writing f first as Φ_i and then as $\Phi_i+\Phi_R$ in (2) and subtracting, the loss due to the obstacle is obtained. The interplay of the physical and the mathematical reasoning is handled in masterly fashion.

D. G. Bourgin (Urbana, Ill.).

Efross, A. M. Transient phenomena in horns. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 82–88 (1940). (Russian. English summary) [MF 4733]

The steady state of sound wave motion in horns is considered. Under certain hypotheses the equation for the potential ϕ becomes

$$\partial^2\phi/\partial x^2+[\partial(\log S)/\partial x][\partial\phi/\partial x]-c^{-2}\partial^2\phi/\partial t^2=0,$$

where $S=S_0(x+x_0)^{n_1}$ and $S=S_0e^{mx}$ in the case of Bessel's and the exponential horn, respectively. The Laplace transform $\phi(x,p)=p\int_0^\infty\exp(pt)\phi(x,t)dt$ can be represented as a sum of two Hankel functions in the first case and as a simple exponential expression in the second. The author considers two kinds of boundary conditions, $v(0,t)=v_0$ and $v(0,t)=a\omega\sin\omega t$, where $v(x,t)$ is the velocity. Formulas for velocity, pressure and for acoustic impedance are given. The solutions are discussed.

S. Bergman.

Tchibissoff, S. Sur la propagation du son dans l'atmosphère et le temps de parcours de l'onde sonore. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 1940, 33–118, 207–222, 475–526 (1940). (Russian. French summary) [MF 2678, 4075, 4076]

The atmosphere at high levels is investigated by means of time measurements in relation to the propagation of sound waves. This paper is devoted to the mathematical theory of sound waves and the mathematical basis for the methods of evaluation of certain phenomena of sound waves. Two problems arise: the direct study of the congruences of rays from a point source, especially the speed of the propagation of the waves and the zones of limited audibility, which depend on the temperature of the air and the wind velocity. The second problem is the development of methods of studying the air and the temperature by means of time measurements of the propagation of sound, and by the study of isochrines on the surface of the earth. Certain hypotheses enable the author to study the propagation of sound as a special case of the propagation of discontinuity surfaces, which are considered as characteristic surfaces. If the speed of the propagation of sound varies continuously with the height, we have, according to Hadamard:

$$uF_x+vF_y+wF_z+c(F_x^2+F_y^2+F_z^2)^{\frac{1}{2}}=1.$$

Using the differential equations of the characteristics we get

$$x=x_0+\int_{z_0}^{z}\frac{c\cos\theta\cos\phi_0+u}{c\sin\theta+w}dz,$$
$$y=y_0+\int_{z_0}^{z}\frac{c\cos\theta\sin\phi_0+v}{c\sin\theta+w}dz,$$
$$t=t_0+\int_{z_0}^{z}\frac{dz}{c\sin\theta+w},$$

where z is the height. Using the principle of Fermat, the equations of the rays can also be put into the form of the Lagrange equations. The author then studies the discontinuity surfaces of order zero in connection with reflections and refractions.

In chapter II the properties of the congruences of rays and the formation of the zones of limited audibility for a continuous "state function" are considered. The case of a point source of sound situated on the earth's surface, the expressions for coördinates, the time and the curves of propagation are studied. Galbrun's classification of rays is also given, and the conditions for the reflection of the rays are found. The point and the moment of return, as well as the time of travel for the second type of rays, are also given. The influence of the wind is studied. The families of surfaces formed by the congruences of rays and their intersection with the surface of the earth are investigated. A classification of these lines is made, and their properties are described. A survey of the various zones of audibility (classified by the scheme of Galbrun) is made. Some examples are discussed. Chapter III is devoted to the study of the surfaces of the hodographs. The author considers the differential properties of these surfaces and the distribution of isoline curves. In chapter IV the condition for the uniqueness of a solution of the inverse problem is studied. In some special cases this solution can be expressed in a closed form.

Under certain conditions a unique solution from hodographs can be determined. A method for an approximate solution of the inverse problem is given, and a numerical example is calculated. *S. Bergman* (Providence, R. I.).

Alvarez Lleras, Jorge. The fundamentals of tropical meteorology. II. Revista Acad. Colombiana Ci. Exact. Fis. Nat. 4, 50–55 (1940). (Spanish) [MF 4527]

The generalized equations of hydrodynamics, which were deduced in chapter I [same Revista 3, 439–447 (1940); cf. these Rev. 2, 268], are here applied to the atmosphere and lead to the general differential equations of atmospheric movement under the conditions prevailing in the tropical zone. The special cases of equilibrium, of purely horizontal motion and of uniform temperature are considered in detail. The meteorological implications and the limitations of the theory are indicated. *P. Nemenyi* (Fort Collins, Colo.).

Tollmien, W. und Schäfer, Manfred. Zur Theorie der Windkanalturbulenz. Z. Angew. Math. Mech. 21, 1–17 (1941).

After a brief review of the present theories of isotropic turbulence and some experimental results, the author attacks the problem of the behavior of a turbulent flow in a wind tunnel downstream from a grid. In this treatment the assumption of isotropy is discarded. The Navier-Stokes differential equations are linearized under the assumption that the fluctuating velocity components u, v and w are small in comparison with the mean stream velocity u_m. The flow can be separated into two parts: (i) a potential motion, (ii) the rotational part, a diffusion flow. It is assumed that the flow is completely known in a certain plane downstream from the grid, designated by $x=0$; the potential flow is then determined by the distribution of pressures in this plane, and the equations for the diffusion flow can be integrated. The author states his belief that this method of integration has not been given in the extensive literature on the linearized equations [e.g., C. W. Oseen, Neuere Methoden und Ergebnisse in der Hydrodynamik, Leipzig, 1927]. It is then shown how the mean values of fluctuating quantities are calculated in terms of their Fourier transforms by use of Parseval's theorem. This method is applied to prove certain theorems of G. I. Taylor and of J. Kampé de Fériet concerning the spectrum of turbulence and the correlation of velocities. Several confirming experimental results are pointed out. Finally the method of Fourier transforms is used to calculate the mean-square values, as functions of x, of the velocity components of the potential and diffusion flows in the wind-tunnel problem. The results show that the potential flow itself cannot be isotropic; in fact, for this flow $u^2=v^2+w^2$. The relationship between the mean-square values of the diffusion components is not deduced. General results regarding the mean values of the mixed products, which occur in the mean-square values of the total velocities, are also left for future investigations. *W. R. Sears* (Inglewood, Calif.).

***v. Mises, R. Some remarks on the laws of turbulent motion in tubes.** Theodore von Kármán Anniversary Volume, pp. 317–327. California Institute of Technology, Pasadena, Calif., 1941.

Considering the mean flow in a circular pipe, the author first assumes the nature of the dependency of the velocity in a region near the wall. He then assumes that there is a region near the center of the pipe where the dependency of the "turbulent viscosity coefficient" (or of the slope of the velocity profile) is known. These assumptions lead to different functional relations for the velocity distribution in a cross section. Following C. B. Millikan [Proc. 5th Int. Cong. App. Mech., 1938, p. 386] he assumes that there exists a region where both relations hold. This yields a functional equation $\varphi(\xi\eta, k)+\psi(\eta, k)=\chi(\xi, k)$ with the condition $\psi(1, k)=0$. This is solved by simple logarithmic expressions for φ, ψ, χ. von Kármán's "logarithmic relation" between the resistance and the Reynolds number is a consequence, but nothing can be deduced regarding the velocity distribution outside of the common region. The results are more precise and more general than those of Millikan, who assumed the central region mentioned above to extend over the whole cross section. For cases of non-circular cross sections the author brings in a third hypothesis: that in a set of similar cross sections the shearing-stress distribution does not depend on the Reynolds number. Results are then obtained that are analogous to those for circular tubes. *W. R. Sears* (Inglewood, Calif.).

Arakawa, Hidetosi. Dissipationstheorem der turbulenten Flüssigkeitsbewegungen. Proc. Phys.-Math. Soc. Japan (3) 23, 535–538 (1941). [MF 5409]

Reynolds' well-known equations for the turbulent motion of an incompressible viscous fluid are used to calculate the rate of change of the kinetic energy. This is given by the sum of two volume integrals involving the mean-velocity components, the viscosity and the "apparent shear stresses." The transformation to cylindrical and spherical coordinates from rectangular Cartesian is shown. *W. R. Sears.*

Kolmogoroff, A. N. On degeneration of isotropic turbulence in an incompressible viscous liquid. C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 538–540 (1941). [MF 4952]

Turbulent flow isotropic in the sense of G. I. Taylor [Proc. Roy. Soc. London. Ser. A. 151, 421–464 (1935)] is considered. For such flows Loitsiansky [Trudii Centr. Aerodin. Inst. Žukovski, no. 440, 24 pp. (1939)] has deduced from von Kármán and Howarth's differential equation [Proc. Roy. Soc. London. Ser. A. 164, 192–215 (1938)] for the propagation of the correlation function $f(r, t)$ that

$$\Lambda=\int_0^\infty bg(r, t)r^4dt$$

does not depend on the time t. (r is the distance from the origin, $g(r, t)$ is the second of the two double correlation functions of von Kármán and Howarth, and b is the mean square of the velocity fluctuations.) The author takes as the "scale of turbulence" the length $L=(\Lambda/b)^{1/5}$. He then defines a Reynolds number $R=Lb^{\frac{1}{2}}/\nu$, where ν is the kinematic viscosity. Following von Kármán and Howarth, he neglects the "triple correlation function" for $R\to 0 (t\to\infty)$; Millionshtchikov's results [C. R. (Doklady) Acad. Sci. URSS (N.S.) 22, 231–233 (1939)] then apply, for example,

$$bf(r, t)=\frac{k}{(4\nu t)^{5/2}}\cdot\exp\left(-\frac{r^2}{8\nu t}\right).$$

For large values of R he assumes with von Kármán [op. cit.] that $f(r, t)=b\psi(r/L)$. He obtains von Kármán's results: $b^{\frac{1}{2}}=u_0(t-t_0)^{-p}$ and $L=L_0(t-t_0)^{1-p}$, where u_0, t_0 and L_0 are arbitrary constants, and moreover deduces that $p=5/7$. *W. R. Sears* (Inglewood, Calif.).

Theory of Elasticity

★Timoshenko, S. The forced vibrations of tie-rods. Theodore von Kármán Anniversary Volume, pp. 226–230. California Institute of Technology, Pasadena, Calif., 1941.

In a hinged tie-rod subjected to small vibrations, the tensile force is assumed to be constant. Then the internal strain energy of bending and of axial tension, as well as the kinetic energy, can be expressed in terms of the generalized time coordinates $q_n(t)$. From Lagrange's equation of motion, $q_n(t)$ is determined in terms of the initial conditions and the generalized exciting force $Q_n(t)$. The case of a point load P moving along the rod at constant velocity v is examined. The critical resonance case occurs when the forced vibration has a period equal to the interval of time in which the disturbing force P travels a distance equal to twice the length of the rod. A special case is that of a flexible wire in which the critical velocity v of the moving load is $v^2/v_0^2=\sigma/E$, where v_0 is the velocity of sound propagation, σ is the tensile stress and E the modulus of th ewire. This critical velocity is sometimes low enough to be encountered in the trolley lines of electric railways and accounts for a heavy impact force at the hinged end as the load passes this hinged end. *D. L. Holl* (Ames, Iowa).

Sakadi, Zyuro. Elastic waves in crystals. Proc. Phys.-Math. Soc. Japan (3) **23**, 539–547 (1941). [MF 5410]

The Christoffel wave equation for plane wave propagation in aeolotropic mediums is applied to the cubic, hexagonal, tetragonal and trigonal crystal systems. In each of the first two systems all classes are identical as far as the elastic behavior is concerned, while there are two groups of classes with slightly different elastic tensors in each of the last two systems. For all six groups the expressions for the six stiffness coefficients which occur in the three-dimensional secular wave matrix are derived in terms of the two spherical angles characterizing the direction of propagation. The "principal directions," that is, those in which the waves are purely longitudinal and transversal or the secular matrix purely diagonal, are obtained by equating the three coupling stiffnesses to zero. The principal directions and the associated triplets of wave velocities are derived for all six crystal groups mentioned. The locations of the principal directions are easily understood in terms of the symmetry characters of the different systems. *H. G. Baerwald.*

Seth, I. D. Reflection and refraction of attenuated waves in semi-infinite elastic solid media. Proc. Indian Acad. Sci., Sect. A. **13**, 151–160 (1941). [MF 4764]

Consider wave systems independent of y in an isotropic elastic solid with boundary $z=0$. On expressing the displacements in terms of a scalar and vector potential we are led to the determination of ψ^i, $i=1, 2, 3$, where $\rho\psi_{tt}^i=[\mu+\delta_1^i(\lambda+\mu)]\nabla^2\psi^i$ subject to four boundary conditions on ψ^1 and ψ^3 and two on ψ^2. The theory of reflection and refraction of plane waves of the form $\exp i\kappa(x\pm\beta^j z-\omega t)$ is classical. The author's contribution is to let $\beta^j=\beta_1^j+i\beta_2^j$. This yields waves with amplitude exponentially dependent on z. Special interest attaches to generation of surface waves. The mathematical analysis for say the ψ^1, ψ^3 waves amounts to substituting the wave form in the boundary conditions. This yields two algebraic equations of the second degree connecting the four components of β^1 and β^3, and two equations involving the wave amplitudes as well. These algebraic equations contain all the implications treated in the paper, such as critical angle of incidence, wave velocity, etc. A similar discussion is given for ψ^2 waves. The results have some application to the theory of earthquake waves. *D. G. Bourgin* (Urbana, Ill.).

Iškov, P. Über die Fortpflanzung elastischer Wellen in einer auf harter Grundlage liegenden Schicht. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 169–176 (1941). (Russian. German summary) [MF 4840]

The author considers the two-dimensional problem of propagation of waves in an infinite horizontal elastic strip occupying the space above a rigid half-plane. It is shown that the lowest value for the velocity of propagation is equal to the velocity of Rayleigh's surface wave. *A. Weinstein* (Toronto, Ont.).

Davila C., Rafael. Elastic waves and the seismological problem. Revista Ci., Lima **43**, 183–204 (1941). (Spanish) [MF 4848]

The author gives an excellent summary in Spanish of the classical theory of elastic body waves in an isotropic solid and in a gravitating compressible sphere. The visco-elastic theory of Sezawa and the theories of Hencky and Uller are briefly mentioned. *J. B. Macelwane* (St. Louis, Mo.).

Garcia, Godofredo. On the propagation of the seismic waves of Mr. C. Somigliana in media having properties of a cubic crystalline system. Revista Ci., Lima **43**, 271–290 (1941) = Actas Acad. Ci. Lima **4**, 53–72 (1941). (Spanish) [MF 5095]

The author calls Somigliana's waves those which correspond to complex roots of the frequency-wave length equation of the well-known Raleigh waves which are propagated along the surface of a homogeneous elastic semi-infinite medium. He investigates plane Somigliana waves for a medium with the strain energy function

$$2W=A_{11}(\epsilon_x^2+\epsilon_y^2+\epsilon_z^2)+B_{11}(\gamma_{xy}^2+\gamma_{xz}^2+\gamma_{yz}^2)+2A_{12}(\epsilon_x\epsilon_y+\epsilon_x\epsilon_z+\epsilon_y\epsilon_z).$$

E. Reissner (Cambridge, Mass.).

Rosenblatt, Alfred. Sur la propagation des ondes sismiques dans les milieux transversalement isotropes. Ondes de M. Somigliana. II. Revista Ci., Lima **43**, 51–73 (1941). [MF 4841]

Weber, Constantin. Über die Minimalsätze der Elastizitätstheorie. Z. Angew. Math. Mech. **21**, 32–42 (1941).

On the basis of the linearized expressions for the strain components the author first establishes the principle of minimum potential energy in the form

$$(*)\qquad \int a(\epsilon,\gamma)dV-\int \bar{P}\cdot\bar{u}dV-\int_{o_p}\bar{P}_s\cdot\bar{u}do_p=\min,$$

and Castigliano's principle in the form

$$(**)\qquad \int b(\sigma,\tau)dV-\int_{o_u}\bar{P}_s\cdot\bar{u}do_u=\min,$$

where a is the strain energy and

$$b(\sigma,\tau)=\sigma_x\epsilon_x+\cdots+\tau_{xy}\gamma_{xy}+\cdots-a(\epsilon,\gamma),$$

with the ϵ and γ eliminated by means of six relations of the form $\sigma_x=\partial a/\partial\epsilon_x$. In (*) the surface integral is to be extended

over the region of prescribed $\bar{P}$ and in (**) over the region of prescribed $\bar{u}$. The author shows that simultaneous application of the two principles in Ritz' fashion may be used to establish inequalities, roughly of the form $f_1(A') \leqq C^*/P^* \leqq f_2(B')$, where, in the example of a beam, C^* is the displacement of the point of load application due to a concentrated load P^*, and A' and B' are the values of $\int a\,dV$ and $\int b\,dV$ which follow from the assumed Ritz' coordinate functions. The only assumption about a which is made is that the medium is "stable," in the sense that the second variation $\delta^2 a$ of the strain energy is positive definite.
E. Reissner (Cambridge, Mass.).

Frola, Eugenio. Sull'elasticità non globalmente lineare. Principii e fondamenti delle teorie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 531–540 (1940). [MF 4785]

By a critical examination of the fundamentals of the ordinary linear theory of elasticity, the author attempts to show that its inadequacy for the analysis of problems of elastic instability is due entirely to the fact that it implies the following assumption: "The first derivatives of the displacement-components are so small that their squares can be neglected compared with the displacement-components themselves." The author suggests an assumption, which, substituted in the system of basic assumptions to the above quoted one, would offer, in his opinion, a solid basis for a general theory of elastic instability. *P. Nemenyi.*

Cicala, Placido. Sulla teoria non lineare di elasticità. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 94–104 (1941). [MF 4794]

A survey of the present status of the general theory of elastic instability, as initiated by the fundamental paper of E. Trefftz [Trans. 3rd Internat. Congr. Applied Mech., Stockholm, 1930], is given with emphasis upon those aspects of this theory which are still in need of further clarification. *P. Nemenyi* (Fort Collins, Colo.).

Cicala, Placido. Sulla stabilità dell'equilibrio elastico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 185–222 (1940). [MF 4776]

Certain simplifications of the general equations of elastic stability are suggested and applied to the problem of stability of thin shells. *P. Nemenyi* (Fort Collins, Colo.).

Viola, Tullio. Nuovi metodi di calcolo per la verifica di un'ala a una speciale forma di instabilità dell'equilibrio elastico. Aerotecnica **20**, 14 pp. (1940). [MF 4988]

The elastic problem of instability is solved by finding the first positive critical value of the velocity V for which the differential equations

$$\frac{d}{dx}\left\{B(x)\frac{d\theta}{dx}\right\} - V^2 Z(x)\frac{d^2y}{dx^2} - V^2 H(x)\theta = 0,$$

$$\frac{d^2}{dx^2}\left\{A(x)\frac{d^2y}{dx^2}\right\} + V^2\frac{d^2}{dx^2}\{Z(x)\theta\} + V^2 K(x)\theta = 0$$

have solutions satisfying the end conditions

$$y = B\frac{d\theta}{dx} = A\frac{d^2y}{dx^2} = \frac{d}{dx}\left(A\frac{d^2y}{dx^2}\right) = 0 \qquad \text{at } x=0,$$

$$\theta = \frac{dy}{dx} = 0 \qquad \text{at } x=l.$$

Some methods of approximation are developed and some numerical results are given. *H. Bateman.*

✶Murnaghan, Francis D. The compressibility of solids under extreme pressures. Theodore von Kármán Anniversary Volume, pp. 121–136. California Institute of Technology, Pasadena, Calif., 1941.

In the first part of this paper the author gives a new derivation of his theory of the finite deformations of an elastic solid [Amer. J. Math. **59**, 235–260 (1937)], this time in matrix instead of in tensor terminology. Specializing his results to the case of an isotropic medium for which the density of elastic energy per unit mass is of the form $\varphi = (\mu + \lambda/2)I_1^2 - 2\mu I_2$, where I_1 is the linear and I_2 the quadratic strain invariant, the author considers in particular the problem of a solid subject to uniform hydrostatic pressure. A relation between pressure and change of volume is obtained which, for the entire pressure range of 50,000 atm., is in very good agreement with experimental results of Bridgman [Proc. Amer. Acad. Arts. Sci. **74**, 21–51 (1940)]. Finally the two-dimensional problem of a "thick" hollow circular cylinder, subject to internal pressure, is considered. By carrying out the first two steps of a successive approximations procedure there is obtained a formula for the deformation which takes into account the finiteness of the displacements. *E. Reissner* (Cambridge, Mass.).

Locatelli, Piero. Estensione del principio di St. Venant a corpi non perfettamente elastici. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 502–510 (1940). [MF 4784]

A new proof of the principle of St. Venant for elastic bodies is given and on this basis an attempt is made to show that the principle holds true also for bodies for which Hooke's law is not valid, provided that the deformations are infinitesimally small and that a deformation-energy exists which is a single-valued function of the configuration.
P. Nemenyi (Fort Collins, Colo.).

Locatelli, Piero. Ancora sul principio di Saint-Venant per corpi non perfettamente elastici. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 125–127 (1941). [MF 4796]

The generalized Saint Venant principle [see the preceding review] is expressed in a different form, particularly adapted to the needs of the theory of structures. *P. Nemenyi.*

Barton, M. V. The circular cylinder with a band of uniform pressure on a finite length of the surface. J. Appl. Mech. **8**, A-97–A-104 (1941). [MF 5146]

The solution to the fundamental problem of a cylinder with a uniform pressure over one half of its length and a uniform tension on the other half is found by using the Papcovitch-Neuber solution to the general equations. The results, given analytically in terms of infinite series, are exhibited as curves giving a complete picture of the stress and deformation. The case of a cylinder with a band of uniform pressure of any length, with the exception of very small ones, is solved by superposition. The stresses and displacements are evaluated for the special cases of a cylinder with a uniform pressure load of 1 diam and ½ diam in length. The problem of a cylinder heated over one half its length is solved by the same means. *M. Sadowsky.*

✶Holl, D. L. Plane-Strain Distribution of Stress in Elastic Media. Iowa Engineering Experiment Station, Bulletin no. 148. Ames, Iowa, 1941. 55 pp.

In the first and second part of this paper the stresses in

a semi-infinite, elastic, isotropic medium in plane strain are determined for normal and tangential surface loads whose intensity diagrams simulate triangles and polygons. In the third part of the paper stresses and surface deflections for a finite layer supported by a rough rigid base are obtained in terms of Fourier integrals. The resultant formulas are evaluated numerically in a great many cases, stress and deflection diagrams are constructed, and certain practically important general conclusions drawn from the results of the calculations. *E. Reissner* (Cambridge, Mass.).

Prager, W. Streamlines and lines of principal stress. Rev. Math. Union Interbalkan. **3**, 63–65 (1941). [MF 4875]

The author shows that the simplest general representation of a plane stress field in an elastic body is in terms of the stress-deviator. Let U, V be the rectangular components of this vector. Introducing the complex quantities $\Omega=U+iV$, $\bar{\Omega}=U-iV$, the author shows that $\partial^2\bar{\Omega}/\partial\bar{z}^2=0$ and therefore $\bar{\Omega}=\bar{z}F(z)+G(z)$ (z and $\bar{z}$ are respectively the complex and conjugate complex coordinate of any point of the plane). The author compares his result with the known hodograph property of any plane irrotational streamfield, which can be written $\partial\bar{\omega}/\partial\bar{z}=0$ or $\bar{\omega}=f(z)$ ($\omega=u+iv$, $\bar{\omega}=u-iv$, u, v being the velocity components). This comparison yields immediately the condition for isomorphy between a field of plane stress and a plane irrotational streamfield, which has been obtained in a different way by Nemenyi [Z. Angew. Math. Mech. 13, 364–366 (1933)]. *P. Nemenyi.*

Gorgidze, A. I. and Ruchadze, A. K. On a numerical solution of integral equations of the plane problem of the theory of elasticity. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 255–258 (1940). (Russian) [MF 5275]

It was shown by N. Muschelišvili [C. R. (Doklady) Acad. Sci. URSS (N.S.) 3, 7–11, 73–77 (1934)] that the solution of the first boundary value problem of the plane theory of elasticity is reducible to the solution of the system of integral equations

$$p(\tau)-\frac{1}{\pi}\int_C[p(t)(1+\cos 2\theta)+q(t)\sin 2\theta]d\theta=A_1(\tau),$$
$$q(\tau)=\frac{1}{\pi}\int_C[p(t)\sin 2\theta+q(t)(1-\cos 2\theta)]d\theta=A_2(\tau),$$

where $A_1(\tau)$ and $A_2(\tau)$ are prescribed functions on the boundary C of the region R within which the solution is described, τ and t are points on the boundary and θ is the angle formed by the vector $\overline{\tau t}$ with the x-axis. The authors indicate a method of numerical solution of these equations, which consists in replacing the integrals by finite sums with the aid of some formula for mechanical quadratures, and thus reducing the problem to a solution of a system of linear algebraic equations. The note contains an estimate for the magnitude of error in an approximate solution.
I. S. Sokolnikoff (Madison, Wis.).

Berg, B. A. Sur le problème à deux dimensions de la théorie d'élasticité pour une bande indéfinie. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 4, 37–74 (1940). (Russian. French summary) [MF 4647]

The paper presents the solution of the first and second boundary value problems of the theory of elasticity for the interior of the region bounded by a pair of parallel lines. The prescribed displacements (first problem) and the prescribed stresses (second problem) are assumed to be representable by Fourier series on the boundary of the region. The final formulas for stresses and displacements are expressed in terms of Fourier coefficients of the boundary value functions. The method of solution depends on the use of Green's function for an infinite strip and follows closely along the lines of the well-known solution by O. Tedone of the corresponding space problems for the region bounded by two parallel planes. *I. S. Sokolnikoff.*

Lekhnitsky, S. G. Plane problem of the theory of elasticity for a medium with a slightly expressed aelotropy. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 433–436 (1941). [MF 4839]

The author considers a plane strain or plane stress problem in an elastic aelotropic medium, that differs very slightly from an isotropic one. It is also assumed that three planes of symmetry exist, so that only four moduli survive. The stresses σ_x, σ_y and τ_{xy} are expressed as second order derivatives of a stress function F which satisfies a generalized biharmonic equation. The solution is obtained by assuming that F is of the form

$$F=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\lambda_1^n\lambda_2^m F_{mn},$$

where $F_{mn}=F_{nm}$ are independent of the parameters λ_1 and λ_2, the latter expressing the degree of departure of the medium from an isotropic medium. These parameters are dimensionless ratios of the moduli and are small in comparison with unity. The ordering of the coefficients of successive powers of these parameters leads to a sequential determination of F_{mn} in terms of two analytic functions $\phi_{ij}(z)$ and $X_{ij}(z)$, where $z=x+iy$. The leading functions ϕ_{00} and X_{00} are determined when the forces on the boundary of the same isotropic region are given and the subsequent ϕ_{ij} and X_{ij} are determined as plane boundary value problems involving the preceding known values. The results show that, when $\lambda_1=\lambda_2=0$, the known values given by Kolosoff and Muschelisvili are obtained. *D. L. Holl.*

Scherman, D. I. Sur un problème de la théorie de l'élasticité. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 907–910 (1940). [MF 3585]

The author considers a domain S, the boundary L of which consists of $(m+1)$ closed curves L_k; S is supposed to be a sum, $S=\sum_{n=0}^{k}S_n$, where every S_n, $n=1,\cdots,k$, is simply connected and lies inside of S. Its boundary is denoted by γ_n. Thus the boundary of S_0 is $L+\sum_{n=1}^{k}\gamma_n$. The paper is devoted to the determination of the elastic state in S, if the exterior forces acting on L and jumps of displacements on γ_k are given. It leads to the following mathematical problem: to determine analytic functions $\phi_n^{(s)}(z)$, $s=1, 2$, defined in S_n and satisfying the following boundary conditions: $P_0=f(t)+C_j$ on L_j ($j=1,\cdots,m+1$), $P_0=P_n$, $M_0=M_n+\overline{g_n(t)}$ on γ_n ($n=1,\cdots,k$). Here $P_n=\overline{\phi_n^{(1)}(t)}+\phi_n^{(2)}(t)+\bar{t}d\phi_n^{(1)}(t)/dt$, $M_n=\kappa\phi_n^{(1)}(t)-\phi_n^{(2)}(t)-\bar{t}d\phi_n^{(1)}(t)/dt$; $f(t)$ and $g_n(t)$ are known functions and κ is the constant of elasticity. It follows that $D_0^{(s)}=D_n^{(s)}$, where

$$D_n^{(s)}=\phi_n^{(s)}-\lim_{z_n\to t}\frac{1}{2\pi i[1+(2-s)\kappa]}\int_{\gamma_n}\frac{g_n^{(s)}(t_1)}{t_1-z_n}dt_1,$$
$$n=1, 2, \cdots, k;\ s=1, 2,$$
$$D_0^{(s)}=\phi_0^{(s)}-\lim_{z_0\to t}\frac{1}{2\pi i[1+(2-s)\kappa]}\int_{\gamma_n}\frac{g_n^{(s)}(t_1)}{t_1-z_0}dt_1$$

and $g_n^{(\nu)}(z)$ are known functions. The author now introduces two functions

$$\phi^{(\nu)}(z)=\phi_0^{(\nu)}(z)-\sum_{n=1}^{k}\frac{1}{2\pi i[1+(2-s)\kappa]}\int_{\gamma_n}\frac{g_n^{(\nu)}(t)}{t-z}dt$$

defined in S_0. By the above relations $\phi^{(\nu)}(z)$ can be analytically continued in S_n. Substituting $\phi^{(\nu)}(z)$ in the initial equations the author obtains integral equations of Fredholm type for the determination of $\phi^{(\nu)}(z)$. The conditions for the existence of a solution are discussed. The case where the constants κ_n of elasticity are different for every S_n is considered. *S. Bergman* (Providence, R. I.).

Scherman, D. Sur les tensions dans une plaque elliptique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 313–314 (1941). [MF 4835]

The author gives a new method for the determination of stresses in an elliptic plate. He uses the integral equation, obtained by Lauricella [Acta Math. 32 (1909)],

$$\omega(t_0)+(2\pi i)^{-1}\int_L \omega(t)d\log[(t-t_0)/(\bar{t}-\bar{t}_0)]$$
$$-(2\pi i)^{-1}\int_L \overline{\omega(t)}d[(t-t_0)/(\bar{t}-\bar{t}_0)]=f(t_0),$$

where L is the boundary of the ellipse and

$$f(t_0)=i\int_0^{t_0}(X_n+iY_n)ds,$$

X_n, Y_n being the components of the exterior forces. Using formulas given by Kolossoff one can express the stresses in terms of $\omega(t)$. In order to find $\omega(t)$ the author transforms the ellipse into a circle of radius ρ by $t=R(\sigma+\sigma^{-1})$. The above integral equation is transformed into a new integral equation. Substituting for $\omega^*(\sigma)=\omega(t)$ the infinite series $\sum_{-\infty}^{\infty}a_k\sigma^k$, the author obtains

$$a_k+a_{-k}\rho^{-4k}+(\rho^4-1)k\bar{a}_k\rho^{-2(k+1)}=A_k,\quad a_k+a_{-k}=-A_k;$$
$$k=1, 2, \cdots,$$

where A_k are known quantities. Thus he can determine the function $\omega^*(\sigma)$ and (using the above results) the stresses. *S. Bergman* (Providence, R. I.).

Weber, C. Halbebene mit Kreisbogenkerbe. Z. Angew. Math. Mech. **20**, 262–270 (1940).

The author deals with the problem of plane stress in a semi-infinite elastic sheet in which a homogeneous tension parallel to the edge of the sheet is modified by the existence of a cut-out at the edge, having the form of part of a circle. The method of solution is a combination of conformal mapping and successive approximations procedure. *E. Reissner* (Cambridge, Mass.).

Tolotti, Carlo. Applicazione di un nuovo metodo di M. Picone all'integrazione delle equazioni dell'elasticità in un parallelepipedo rettangolo. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 514–524 (1940). [MF 4993]

The author applies the finite Fourier transform to the integration of the linearized equations of elasticity under prescribed boundary conditions. The boundary of the region considered is a rectangular parallelepiped and the body is homogeneous and isotropic. An application of the finite Fourier transform over two independent variables to the linearized equations of elasticity results in a system of ordinary linear inhomogeneous differential equations. This system is integrated by classical methods. The superfluous boundary elements which are introduced by the transformation are eliminated by applying the regularity condition imposed upon the finite Fourier transform. *A. E. Heins* (Lafayette, Ind.).

Friedmann, M. M. Über einige Fragen der Theorie der Biegung von dünnen isotropen Platten. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, no. 1, 93–102 (1941). (Russian. German summary) [MF 4658]

The author utilizes the method of solution of plane problems of the theory of elasticity developed by N. I. Muschelisvili to determine the deflection of isotropic plates under a prescribed distribution of forces and moments applied along the boundary of the plate. The following cases are considered: (a) circular plate, (b) an infinite plate with circular opening, (c) plate bounded by two concentric circles. *I. S. Sokolnikoff* (Madison, Wis.).

*Donnell, L. H. **Stress concentrations due to elliptical discontinuities in plates under edge forces.** Theodore von Kármán Anniversary Volume, pp. 293–309. California Institute of Technology, Pasadena, Calif., 1941.

Stress distributions are found for an infinite plate, under any uniform edge forces, containing an elliptical region filled with material having a stiffness K times that of the plate. $K=0$ gives the case of an elliptical hole, $K=1$ a simple plate without discontinuity, while $K>1$ gives the case of reinforcements. Thus by varying K and the proportions of the elliptical region, a relatively complete picture is afforded of the general effects of discontinuities of different kinds and shapes. The case of reinforcements, which has received little attention previously, is shown to be of importance. *M. A. Sadowsky* (Chicago, Ill.).

*Synge, J. L. and Chien, W. Z. **The intrinsic theory of elastic shells and plates.** Theodore von Kármán Anniversary Volume, pp. 103–120. University of California Press, Berkeley, Calif., 1941.

The object of the paper is to provide a concise formulation of the basic equations of the theory of shells that can be used as a foundation for the approximate theory of thin plates and shells. After introducing the notions of macroscopic stress and bending tensors (that is, the mean values of stresses across the thickness of the shell), the authors derive six tensor equations of static equilibrium. The equations thus obtained are free of any assumptions regarding the thickness of the shell. The authors develop the geometric theory of strains in the interior of the shell (microscopic theory) and obtain the equations of compatibility. When the microscopic strains are connected with the microscopic stresses via Hooke's law for an isotropic body, one can deduce the macroscopic stresses by integration over the thickness of the shell. *I. S. Sokolnikoff* (Madison, Wis.).

Lechnizky, S. G. Biegung nicht homogener anisotroper symmetrisch aufgebauter dünner Platten. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, no. 1, 71–92 (1941). (Russian. German summary) [MF 4657]

The author gives a derivation of the equation of bending of plates consisting of isotropic or anisotropic layers and of plates for which the elastic constants are continuous even

functions of the thickness. The equation in both cases is

$$g_{11}\frac{\partial^4 w}{\partial x^4}+4g_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(g_{12}+2g_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+4g_{26}\frac{\partial^4 w}{\partial x\partial y^3}+g_{22}\frac{\partial^4 w}{\partial y^4}=\frac{3q}{2h^3},$$

where the g_{ik} are functions of the elastic constants.
A. Weinstein (Toronto, Ont.).

Federhofer, Karl. Berechnung der kleinsten Knicklast einer schwach verjüngten oder verdickten Kreisringplatte. Akad. Wiss. Wien, S.-B. IIa. **149**, 59–75 (1940). [MF 4935]

In a previous paper [Ing.-Arch. **11**, 224–238 (1940); these Rev. **2**, 174] the author developed the general equations for the buckling of a plate in the form of a circular ring and of variable thickness. Except for two laws of variation of the thickness, the solution of the problem could be accomplished only with the aid of infinite series and consequent long numerical calculations. In the present paper in which the thickness of the plate is taken to vary only slightly the approximate method of Galerkin is used. The approximate form of the buckled surface is assumed to be that into which the corresponding plate of constant thickness would buckle. The Ritz method is also employed. The results of the two methods are compared. *H. W. March.*

***Friedrichs, K. O. On the minimum buckling load for spherical shells.** Theodore von Kármán Anniversary Volume, pp. 258–272. California Institute of Technology, Pasadena, Calif., 1941.

The author discusses the influence of certain simplifying assumptions introduced by von Kármán and Tsien in the non-linear theory of the minimum buckling load [von Kármán and Tsien, J. Aeronaut. Sci. **7**, 43–50 (1939); these Rev. **2**, 175]. Modifications of these assumptions are proposed and a somewhat different procedure for obtaining the buckling load is indicated. The assumption of buckling in the neighborhood of a point proposed by von Kármán and Tsien is formulated as a boundary layer phenomenon. The assumption of rotational symmetry of the deflection is retained but the assumptions that the deflection is parallel to the axis of symmetry and that Poisson's ratio vanishes are removed. *A. E. Heins* (Lafayette, Ind.).

Biezeno, C. B. and Koch, J. J. Some explicit formulae, of use in the calculation of arbitrarily loaded, thin-walled cylinders. Nederl. Akad. Wetensch., Proc. **44**, 505–512 (1941). [MF 4974]

It is known that the equations of the theory of bending and stretching of thin circular cylindrical shells can be integrated by means of trigonometric double series, the coefficients in these series depending on the corresponding coefficients in the series for the load terms. [See H. Reissner, Z. Angew. Math. Mech. **13** (1933); W. Flügge, Statik und Dynamik der Schalen, Berlin, 1934; S. Timoshenko, Theory of Plates and Shells, New York, 1940.] In the present paper the authors give general explicit formulas for the coefficients of the various series for displacements and stress resultant components. *E. Reissner* (Cambridge, Mass.).

Kiltchevsky, N. Les équations fondamentales de l'équilibre des enveloppes élastiques et quelques méthodes de leurs intégration. III. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1941**, no. 6, 51–105 (1941). (Ukrainian. Russian and French summaries) [MF 4499]

[Parts I and II appeared in the same Rec. Trav. **1940**, nos. 4 and 5; cf. these Rev. **2**, 172.] In this part of the paper devoted to a systematic treatment of the theory of shells, the author reduces the differential equations of the boundary value problems treated in parts I and II to a system of integral equations. The paper contains a discussion of some approximate methods of solution of these equations, and provides a proof of convergence of the process of obtaining an approximate solution that is based on a method of least squares. Some attention is given to the question of existence and uniqueness of solution of the boundary value problems of the theory of shells.
I. S. Sokolnikoff (Madison, Wis.).

Marguerre, K. Spannungen in Ausschnittversteifungen. Luftfahrtforschung **18**, 253–261 (1941). [MF 5475]

The author determines the distribution of internal moments for an elastic ring, the centerline of which is the curve of intersection of two circular cylinders with intersecting, mutually perpendicular axes. The loading conditions which are considered are those for the stiffening rings of intersecting shells when (1) both shells are subjected to uniform internal pressure, (2) one of the shells is subjected to uniform axial tension, (3) one of the shells is subjected to a state of uniform shear stress. The influence of the deformation of the ring on the state of stress in the shells (and therefore on the loads applied to the ring) is neglected. Explicit expressions are given for the two bending moment components and for the twisting moment, in their dependence on the ratio of the radii of the intersecting cylinders and on the cross-sectional characteristics of the ring. Comprehensive numerical results are given in the form of diagrams and simple approximate formulas are derived for the "design" moments. *E. Reissner* (Cambridge, Mass.).

***Arnstein, Karl. The engineering treatment of ring or wheel problems.** Theodore von Kármán Anniversary Volume, pp. 195–211. California Institute of Technology, Pasadena, Calif., 1941.

This paper deals with the various mathematical treatments of ring problems, extended to include refinements necessary for lightweight structures. The analysis is indicated by three different methods: (a) least work, (b) differential equations of equilibrium, (c) difference equations of equilibrium. *A. E. Heins* (Lafayette, Ind.).

Newing, S. T. Determination of the shearing stresses in axially symmetrical shafts under torsion by finite difference methods. Philos. Mag. (7) **32**, 33–49 (1941). [MF 5089]

The author applies the method of finite differences to the solution of the torsion problem for three different types of symmetrical shafts. While methods for the solution by difference equations for similar problems are available for finite regions [see L. F. Richardson, Philos. Trans. Roy. Soc. (A) **210**, 307–357 (1910)], such problems described above require the solution over infinite regions. A method of iteration is employed and the convergence of the method is discussed. *A. E. Heins* (West Lafayette, Ind.).

***Westergaard, H. M. On the elastic distortion of a cylindrical hole by a localized hydrostatic pressure.** Theodore von Kármán Anniversary Volume, pp. 154–161. California Institute of Technology, Pasadena, Calif., 1941.

A hydrostatic pressure is applied within a small part only of the length of a cylindrical hole in a large elastic solid. An approximate analysis, utilizing as a guide the behavior of a semi-infinite solid in the neighborhood of a discontinuity in surface pressure, is employed to obtain an approximate expression for the radial displacement. It is found that the maximum radial displacement is smaller than if the pressure were applied over the entire length of the hole.

H. W. March (Madison, Wis.).

Papkovitch, P. F. On the deformation of prismatic bars. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 4, 27–36 (1940). (Russian. English summary) [MF 4646]

An exposition of some general and formal mathematical ideas on the integration of differential equations of elasticity in case of prismatic bars. In the concluding section the author expresses doubts as to any practical applicability of his methods. *M. A. Sadowsky* (Chicago, Ill.).

Hetényi, Miklós. On similarities between stress and flow patterns. J. Appl. Phys. 12, 592–595 (1941).

Summary and review of the literature regarding analogies and relationships between plane states of stress and plane states of fluid flow. *E. Reissner* (Cambridge, Mass.).

Prager, W. A new mathematical theory of plasticity. Rev. Fac. Sci. Univ. Istanbul (A) **5**, 215–226 (1941). (English. Turkish summary) [MF 4955]

The simplest amongst the stress strain relations suggested by the author in an earlier publication [Proc. 5th Internat. Congr. Applied Mech., Cambridge, Mass., 1938, p. 234 ff.] is repeated and taken as the basis for a new theory of plasticity. The details of the theory are worked out only for the case of plane strain with particular reference to problems showing a cylindrical symmetry, such as rotation of a rigid cylinder in an infinite plastic mass and a thick-walled tube under internal pressure; from this latter problem the way is shown for the discussion of the pure flexure of a sector of a circular ring. *P. Nemenyi.*

Sokolowsky, W. Über den Druck eines plastischen Kontinuums auf einem harten Stempel. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 5–6, 19–34 (1940). (Russian. German summary) [MF 4650]

The author considers a hard stamp which presses on a plastic medium, with friction between the stamp and the medium. Using the equations for the plane plastic stresses [see Hencky, Z. Angew. Math. Mech. **3**, 241–251 (1923)], he considers the case where the boundary is a straight line. Three possibilities are discussed: $\tau_{nt}=0$, $\tau_{nt}=\text{const.}$, $\tau_{nt}=\sigma_n \operatorname{tg} \mu$, where τ_{nt} are the tangential stresses along the contact line and μ is the angle of friction. Some of the author's results were obtained earlier by Prandtl [Z. Angew. Math. Mech. **3**, 401–406 (1923)]. The author also studies the case where the plastic medium has a curvilinear boundary. Using the results of Christianowitch [Rec. Math. [Mat. Sbornik] **1** (43), 511–534 (1936)] and of the author's previous papers [C. R. (Doklady) Acad. Sci. USSR (N.S.) **22**, 153–157 (1939); **23**, 15–17 (1939)] he determines the normal stresses along the line of contact and the force necessary in order that the stamp enters a given depth.

S. Bergmann (Providence, R. I.).

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Cf. the bibliographical note in these Rev. **1**, 32. The present issue contains reprints of five additional papers of Aureliano de Mira Fernandes published in the Rendiconti della R. Accademia dei Lincei between 1933 and 1937.

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These volumes contain papers published in various Japanese journals during 1939–1940.